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TABLE OF CONTENTS

History 197
Foundations 198
Algebra 200
Abstract algebra 201
Theory of groups 203
Number theory 207
Analysis 214
Calculus 215
Theory of sets, theory of functions of real variables 216
Theory of functions of complex variables . 220
Theory of series 226
Fourier series and generalizations, integral transforms 228
Polynomials, polynomial approximations . 232
Special functions 233
Harmonic functions, potential theory . . 234
Differential equations 236
Difference equations, special functional equations 245
Integral equations 247
Functional analysis, ergodic theory . . . 249
Calculus of variations 257
Theory of probability 258
Mathematical statistics 259
Mathematical economics 261
Mathematical biology 263
Topology 264
Geometry 268
Convex domains, extremal problems, integral geometry 270
Algebraic geometry 272
Differential geometry 274
Numerical and graphical methods . . . 282
Relativity 289
Mechanics 290
Hydrodynamics, aerodynamics, acoustics . 292
Elasticity, plasticity 299
Mathematical physics 303
Optics, electromagnetic theory 303
Quantum mechanics 306
Thermodynamics, statistical mechanics . . 307

AUTHOR INDEX

Abramyan, B. L.-Dărbaĭyan, M. M. 302
Aczél, J. 246
Aczél, J.-Kalmár, L.-Mikusiński, J. G. 246
Adauto da Justa Medeiros, L. 226
Agnew, R. P. 226
Aleksandrov, P. S.-Urysohn, P. S. 264
Alexiewicz, A. 250
Alexiewicz, A.-Orlicz, W. 250
Allen, D. N. de G.-Severn, R. T. 287
Amitsur, S. 282
Andelić, T. P. 215
Andrejewski, W. 304
Ankeny, N. C.-Artin, E.-Chowla, S. 212
Anós y Díaz de Arcaya, A. See Azorín Poh, F.
Arens, R.-Dugundji, J. 264
Arnold, B. H. 216
Artin, E. See Ankeny, N. C.
Artobolevskiĭ, I. I. 289
Ascoli, G. 242
Athen, H. 215, 288
Auner, M. 233
Austin, M. C. 226
Avakumović, V. G. 237
Azorín Poh, F.-Ríos García, S.-Casado Maceda, E.-Anós y Díaz de Arcaya, A. 259
Bagemihl, F. 221
Banach, S. 216
Banachiewicz, T. 285, 288
Baranov, G. G. 294
Bari, N. K. 229
Bari, N. K.-Lyusternik, L. A. 198
Barnard, G. A. 260
Barnes, R. C. M.-Cooke-Yarborough, E. H.-Thomas, D. G. A. 289
Bartsch, H. 277
Bazarov, I. P. See Vlasov, A. A.
Beeger, N. G. W. H. 211
Bell, P. O. 276, 277
Belyakova, V. K. 293
Benedicty, M. 272
Bereis, R. 292
Berezman, A. M. 278
Bergström, H. 258
Berlin, T. H. See Montroll, E. W.
Bers, L. 244, 245
Beurling, A. 230
Bieberbach, L. 286
Bielecki, A. 237
Bilo, J. 269
Bing, R. H. 264, 265
Bing, R. H.-Floyd, E. E. 265
Birkhoff, G.-Young, D. M.-Zarantonello, E. H. 288
Blackwell, D. 260
Blank, A. See Keller, J. B.
Blanuša, D. 289
Blaschke, W. 274
Bleksley, A. E. H. 290
Bloch, I.-Hull, M. H., Jr.-Broyles, A. A.-Bouricius, W. G.-Freeman, B. E.-Breit, G. 234
Bojanić, R. 214
Bompiani, E. 276
Boothby, W. M. 266
Bopp, F.-Haag, R. 239
Born, M. 199
Bouligand, G. 269
Bouricius, W. G. See Bloch, I.
Brahana, H. R. 198
Brauer, A. 211
Breit, G. See Bloch, I.
Breit, G.-Hull, M. H., Jr. 234
Bremmer, H. 303
Brewer, B. W. 208
Brock, J. E. 287
Brown, G. W. 261
Broyles, A. A. See Bloch, I.
de Bruijn, N. G.-van Ebbenhorst Tengbergen, C.-Kruyswijk, D. 207
Bückner, H. 288, 289
Buehler, R. J.-Hirschfelder, J. O. 236
Bugaec, P. T. 229
Burger, E. 204, 267
Burkill, H. 230
Butlewski, Z. 237
Byuilgens, S. S. 292
Campbell, J. P.-McKinney, M. O. 298
Casado Maceda, E. See Azorín Poh, F.
Čarnyĭ, I. A. See Vlasov, I. O.
Carrier, G. F. See Munk, W. H.
Carrière, P. 298
Cavallaro, V. G. 268
Cesari, L. 220
Chadaya, F. G. 302
Chaki, M. C. 282
Chandrasekhar, S.-Münch, G. 249
Chandrasekharan, K. 198
Chapman, D. G. 260
Chapman, D. R. 295
Charles, H. 239, 244
Châtelet, A. 211
Chazy, J. 198
Cheng, Che-Min 301
Chowla, S. See Ankeny, N. C.
Cicala, P. 298
Citlanadze, È. S. 251
Claf, L. Ya. See Kuznecov, P. I.
Clark, R. S. 279
Clemmow, P. C. 215
Cocchi, G. 294
Cohen, I. S.-Kaplansky, I. 292
Cohen, R. S.-Spitzer, L., Jr.-Routly, P. McR. 307
Cohn, H. 212
Collatz, L. 285
Conte, L. 197
Conti, R. 243
Conway, A. W. 197
Cooke-Yarborough, E. H. See Barnes, R. C. M.
Coolidge, J. L. 197
Corpaciu, A. J. 288
van der Corput, J. G. 202
Couffignal, L. 284
Courbon, J. See Robinson, J.-R.
Court, L. M. 215
Cuesta, N. 216
Curtiss, J. H. 286
Dantzig, G. B. 262
Dantzig, G. B. See Wood, M. K.
Darling, D. A. 258
Davis, R. C. 259
Davison, B.-Kushneriuk, S. A.-Seidel, W. P. 248
De Donder, T. 263
Delone, B. N. 198
Dénes, P. 212
Depman, I. Ya. 197
Devidé, V. 201
De Vogelaere, R. 304
Díaz, J. B. 235
Dieudonné, J. 205, 218
Dijksterhuis, E. J. 197

(*Continued on cover 3*)

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HISTORY

Inamdar, M. G. A formula of Bhaskara for the chord of a circle leading to a formula for evaluating $\sin \alpha^\circ$. Math. Student **18** (1950), 9–11 (1951).

This paper considers a rational approximation, given by Bhāskara in his Līlāvatī, for the sine of an angle and a modification of this formula by the Hindu astronomer Gaṇeśa. A comparison of results by these approximations with modern results shows a good agreement.

E. B. Allen (Troy, N. Y.).

Krishnaswami Ayyangar, A. A. Remarks on Bhaskara's approximation to the sine of an angle. Math. Student **18** (1950), 12 (1951).

The author points out that Bhāskara's rational approximation for the sine of an angle [see the preceding review] is the best that can be devised. *E. B. Allen.*

Somayaji, D. A. Bhaskara's calculations of the gnomon's shadow. Math. Student **18** (1950), 1–8 (1951).

This is a discussion of two problems considered by Bhāskara in his Siddhāntaśiromaṇi: (i) to find the gnomon's shadow in a given direction on any day in a place of known latitude; (ii) to draw the correct East-West line by marking two gnomon's shadows on the same day. Bhāskara's proof and one by modern methods are given for each problem. It is noted that the first problem implies a result in spherical trigonometry and the second implies a use of finite differences. Certain fundamental Hindu latitudinal triangles are specified. *E. B. Allen* (Troy, N. Y.).

Dijksterhuis, E. J. Deux traductions de Proclus. Arch. Internat. Hist. Sci. (N.S.) **4**, 602–619 (1951).

Coolidge, J. L. The story of tangents. Amer. Math. Monthly **58**, 449–462 (1951).

Koyré, Alexandre. La gravitation universelle de Kepler à Newton. Arch. Internat. Hist. Sci. (N.S.) **4**, 638–653 (1951).

Norden, A. P. 125 years of non-Euclidean geometry. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 3–9 (1 plate) (1951). (Russian)

Godeaux, Lucien. Les recherches mathématiques en Belgique dans ces dernières années. Bull. Soc. Math. Belgique **3** (1949–1950), 32–40 (1951).

***Schwartz, L. Les mathématiques en France pendant et après la guerre.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 49–67. University of Toronto Press, Toronto, 1951. $6.00.

***Zygmund, A. Polish mathematics between the two wars (1919–39).** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 3–9. University of Toronto Press, Toronto, 1951. $6.00.

Depman, I. Ya. M. F. Bartel's, a teacher of N. I. Lobačevskiĭ. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **3**, 475–485 (1950). (Russian)

Conte, Luigi. I fratelli Bernoulli e la plurisezione degli archi. Period. Mat. (4) **29**, 113–126 (1951).

Pelseneer, J. Une lettre inédite de Cauchy. Arch. Internat. Hist. Sci. (N.S.) **4**, 631–633 (1951).

Taton, René. Documents nouveaux concernant Desargues. Arch. Internat. Hist. Sci. (N.S.) **4**, 620–630 (8 plates) (1951).

Godeaux, Lucien. Obituary: Federigo Enriques (1871–1946). Bull. Soc. Roy. Sci. Liège **20**, 77–85 (1951).

Pelseneer, Jean. Une lettre inédite d'Euler à Rameau. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 480–482 (1951).

Sokolovskiĭ, V. V. On the life and scientific activity of academician B. G. Galerkin. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1951**, 1159–1164 (1 plate) (1951). (Russian)

Kolmogorov, A. N. The work of I. M. Gel'fand in algebraic questions of functional analysis. Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 184–186 (1951). (Russian)

A discussion of the work for which Gel'fand won a Stalin prize.

Ippolit Stepanovič Gromeka (on the centenary of his birth). Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 393–395 (1 plate) (1951). (Russian)

The scientific work of I. S. Gromeka. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 396–408 (1951). (Russian)

The discussion of Gromeka's scientific work is followed by a list of his published works and a list of biographical material.

✓***Conway, A. W. Hamilton, his life, work, and influence.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 32–41. University of Toronto Press, Toronto, 1951. $6.00.

Vollgraff, J. A. Deux lettres de Christiaan Huygens. Arch. Internat. Hist. Sci. (N.S.) **4**, 634–637 (1951).

Laptev, B. L. The life and work of N. I. Lobačevskiĭ. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 10–17 (1951). (Russian)

Rybkin, G. F. On the Weltanschauung of N. I. Lobačevskiĭ. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 18–30 (1951). (Russian)

Bari, N. K., and Lyusternik, L. A. The work of D. E. Menšov on trigonometric series. Uspehi Matem. Nauk (N.S.) 6, no. 4(44), 187–189 (1951). (Russian)

A discussion of the work for which Menšov won a Stalin prize.

Brahana, H. R. Obituary: George Abram Miller. Amer. Math. Monthly 58, 447–449 (1951).

Ivan Georgievič Petrovskiĭ (for his 50th birthday). Uspehi Matem. Nauk (N.S.) 6, no. 3(43), 160–164 (1 plate) (1951). (Russian)

A list of his published mathematical papers is included.

Chandrasekharan, K. Obituary: S. S. Pillai. J. Indian Math. Soc. (N.S.) Part A. 15, 1–10 (1 plate) (1951).

A list of Pillai's published mathematical work is included. He was born April 5, 1901 and died August 31, 1950.

Chazy, Jean. Henri Poincaré et la mécanique céleste. Bull. Astr. (2) 16, 145–160 (1951).

Dugas, René. Henri Poincaré devant les principes de la mécanique. Revue Sci. 89, 75–82 (1951).

Pauling, Linus. Obituary: Arnold Sommerfeld: 1868–1951. Science (N.S.) 114, 383–384 (1951).

Heisenberg, W. Arnold Sommerfeld. Naturwissenschaften 38, 337–338 (1951).

Kolmogorov, A. N., and Hinčin, A. Ya. The work of N. V. Smirnov on the investigation of properties of variational series and on nonparametric problems of mathematical statistics. Uspehi Matem. Nauk (N.S.) 6, no. 4(44), 190–192 (1951). (Russian)

Discussion of the work for which Smirnov won a Stalin prize.

***Predvoditelev, A. S. Nikolaĭ Alekseevič Umov, 1846–1915.** Izdat. Moskov. Gosudarstv. Univ., Moscow, 1950. 53 pp.

A list of Umov's published works as well as a bibliography of biographical material is included.

Delone, B. N. On the sixtieth birthday of Ivan Matveevič Vinogradov. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 385–394 (1 plate) (1951). (Russian)

A list of Vinogradov's published works is included.

Hristianovič, S. A. The scientific legacy of N. E. Žukovskiĭ. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1951, 1137–1151 (2 plates) (1951). (Russian)

Golubev, V. V. N. E. Žukovskiĭ's work in aerodynamics. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1951, 1152–1158 (1951). (Russian)

FOUNDATIONS

Greniewski, Henryk. Functors of the propositional calculus. Ann. Soc. Polon. Math. 22, supplement, 78–86 (1950).

It is shown that the following devices can be defined in the "language" of propositional algebra: 0, 1, definition by truth tables, propositional quantification, variable functors, duality. Some of the devices in question are those characterizing the "protothetic" of Leśniewski. It follows that the language of this protothetic is not more expressive than that of ordinary propositional algebra (i.e. two-valued algebra with a single category of variables, no quantifiers or constants, and the usual functors). *H. B. Curry.*

Greniewski, Henryk. Certain notions of the theory of numbers as applied to the propositional calculus. Časopis Pěst. Mat. Fys. 74 (1949), 132–136 (1950). (English. Polish summary)

This is a partial formalization of certain considerations belonging to the higher theory (which may be variously regarded, depending on ones point of view, as the "syntax", the "meta theory", the "epitheory" etc.) of the ordinary propositional algebra. The considerations are those in which we prove (epi- or meta-) theorems involving expansions, normal forms, etc., of functions involving an unspecified number of arguments, using letter subscripts which can take unspecified numerical values. Ordinarily we use these numerical indices intuitively. The author gives definitions which would enable one to incorporate some of this theory in a formalism combining propositional algebra, recursive arithmetic, and, in some cases, quantification with respect to propositional variables. Some of these definitions involve use of the greatest integer function, i.e., that function $[x]$ of x whose value is the greatest integer n such that $n \leqq x$. A number of simple theorems are stated without proof. Reference is made to the paper reviewed above.

H. B. Curry (State College, Pa.).

Wang, Hao. Arithmetic translations of axiom systems. Trans. Amer. Math. Soc. 71, 283–293 (1951).

A mapping of a formal system S into a system S' is called a translation if it carries (i) statements into statements, (ii) negation into negation, (iii) theorems of S into theorems of S'. The translation is supposed to be effective, i.e., such that in some Gödel numbering it is given by a general recursive function. Let $\mathrm{Con}(S)$ denote the arithmetical statement expressing the consistency of S, Z the system of number theory as formalized by Hilbert and Bernays [Grundlagen der Mathematik, Band I, p. 371, Springer, Berlin, 1934], Z_S the system obtained from Z by adding $\mathrm{Con}(S)$ as an axiom. If Z_S is consistent, then S is consistent; if S is consistent and Z is ω-consistent, then Z_S is consistent. The main theorem is: Every system S is translatable into Z_S. The proof is only sketched; it consists in an arithmetization of Henkin's proof for the completeness of the first order predicate calculus [J. Symbolic Logic 14, 159–166 (1949); these Rev. 11, 487]. Applications are made to proofs of relative consistency. Let N be a system of set theory in which the distinction between sets and classes (the latter not being elements of sets or classes) is made. Let N' be the system obtained from N by the introduction of class variables with corresponding quantifiers, and $N\sharp$ the system obtained from N by adding $\mathrm{Con}(N)$ as an axiom. N is translatable into Z_N; this translation can be extended to a translation of N' into $N\sharp$. If N is ω-consistent, then $N\sharp$, hence also N', is consistent. Finally, a sequence of systems $L_0 = Z, L_1, L_2, \cdots$ is defined such that, if they are all con-

sistent, L_n is translatable into L_m if and only if $n \leqq m$; there exists also a sequence of predicates $P_1, P_2, \cdots$, such that P_n is definable in L_m if and only if $n \leqq m$. *A. Heyting.*

Rose, Alan. Remarque sur les notions d'indépendance et de non-contradiction. C. R. Acad. Sci. Paris **233**, 512–513 (1951).

This note gives an example of a logistic system E and a formula $\mathfrak{A}$ such that $\mathfrak{A}$ cannot be deduced in E, but nevertheless when $\sim\mathfrak{A}$ is adjoined to the axioms of E the system becomes inconsistent. The axioms of E are $((X\rightarrow\sim X)\rightarrow X)\rightarrow X$, $X\rightarrow(Y\rightarrow X)$, $\sim Y\rightarrow\sim X\rightarrow(X\rightarrow Y)$, $(X\rightarrow Y)\rightarrow((Y\rightarrow Z)\rightarrow(X\rightarrow Z))$ with modus ponens and substitution as the rules of inference. $\mathfrak{A}$ is the formula $(\sim X\rightarrow X)\rightarrow X$. *I. L. Novak* (Princeton, N. J.).

Sobociński, Bolesław. L'analyse de l'antinomie russellienne par Leśniewski. IV. Methodos **2**, 237–257 (1950).

This is the fourth and (apparently) final installment of the author's exposition of the explanation given by Leśniewski for the Russell paradox. The previous installments [Methodos **1**, 94–107, 220–228, 308–316 (1949); these Rev. **11**, 73, 412, 708] were obscured by the fact that, although the author presupposes a logical basis comprehensible to everyone who understands the Peano-Russell symbolism, he nevertheless uses "ϵ" and some other symbols in an inadequately explained sense; this sense appears to be that in Leśniewski's papers, which are comparatively inaccessible (and are not accessible to the reviewer). This installment, which gives the final explanation of the paradox, makes the whole investigation somewhat clearer. The formalism assumed in the previous installments has the following characteristics: there are two kinds of terms which will be called here subjects and predicates; the elementary statements are of the form $A\epsilon a$, where A is a subject and a is a predicate; every subject is also a predicate, so that $A\epsilon A$ is significant; and there is an operation forming from a predicate a a predicate $Kl(a)$. If one assumes the schemes

C1: $(\exists B)B\epsilon a.\supset.(\exists A).A\epsilon Kl(a)$

A2: (A,B,a,b): $A\epsilon Kl(a).A\epsilon Kl(b).B\epsilon b.\supset.B\epsilon a$,

together with the rules of the elementary functional calculus, certain principles connecting with equality, and the existence of at least two objects, one has a contradiction; one also gets a contradiction in certain analogous ways. In this installment the author explains the paradox by saying that A2 is intuitively false under either of two possible interpretations of $Kl(a)$. In the first interpretation, $A\epsilon Kl(a)$ is equivalent to $A\epsilon a$, and A2 is clearly false. In the second interpretation, $Kl(a)$ is the whole formed by all the a's; then if the a's are the books on a certain table and the b's the pages of those books, $Kl(a)$ and $Kl(b)$ are the same whole, but a page is not one of the books. Thus Leśniewski denies A2, and the paradox disappears. In the second part of the installment the ordinary formulations of the paradox are considered. It is argued that the paradox comes either from violating the principle of semantical categories—a form of the theory of types—or from improperly constructed definitions. *H. B. Curry* (State College, Pa.).

Saarnio, Uuno. Ueber den Begriff der Existenz. Methodos **2**, 306–328 (1950).

The author stresses the importance of the distinction between two conceptions of existence, namely, (1) existence based on experience and (2) existence based on an inventory. An inventory has to answer the following requirements: (i) every object inserted obtains a definite number, different objects obtain different numbers; (ii) no part or element of an inserted object is inserted; (iii) if certain objects have been inserted, no aggregate or collection constituted by these objects is inserted. The notion of an inventory being thus explained, the notion of existence can be defined as follows: those objects do exist, which can be inserted in one and the same inventory. Besides the existing there is all that can be constituted on the basis of the existing: element, part, set, and relation with regard to the existing. *E. W. Beth* (Amsterdam).

Markov, A. A. On the representation of recursive functions. Amer. Math. Soc. Translation no. **54**, 13 pp. (1951).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 417–424 (1949); these Rev. **11**, 151.

Fogelis, E. On finite proofs of arithmetical theorems. Latvijas PSR Zinātņu Akad. Vēstis **1950**, no. 6 (35), 81–86 (1950). (Latvian. Russian summary)

An exposition is given of the concept of a constructive formulation in mathematics, and the definition of finiteness of proofs is given. The principal difference between analytical and finite proofs of arithmetical theorems is that for finite proofs, in contrast to analytic ones, the axiom of continuity is not required. *Author's summary.*

Germansky, Baruch. An alternative proof of a theorem of equivalence concerning axioms of natural numbers. Riveon Lematematika **4**, 18–21 (1950). (Hebrew. English summary)

Born, M. Physics and metaphysics. Mem. Proc. Manchester Lit. Philos. Soc. **91** (1949–50), 35–53 (1951).

***Eddington, Arthur. The Philosophy of Physical Science.** Cambridge, at the University Press, 1949. ix+230 pp. $3.00.

Reprint of the 1939 edition.

Fleischmann, R. Die Struktur des physikalischen Begriffssystems. Z. Physik **129**, 377–400 (1951).

It is pointed out that the "dimensions" of physical quantities occurring in any physical theory form an Abelian group. In many cases, it is natural to restrict attention to integral power-products of basic dimensions; this gives a free Abelian subgroup, in which a set of elements forms an integral "basis" if and only if the determinant of its exponents is unity. *G. Birkhoff* (Cambridge, Mass.).

Matthieu, P. Die Rolle der Analogien in der angewandten Mathematik. Vierteljschr. Naturforsch. Ges. Zürich **96**, 103–119 (1951).

This is the author's inaugural lecture at the Eidgenössische Technische Hochschule of Zurich. The author calls two entities analogous if they can be considered as different interpretations of one and the same abstract relationship or set of relationships. Various analogies between different geometries are pointed out on the basis of Felix Klein's Erlangen lecture. In the field of mechanics Mohr's analogy between a beam and a funicular curve, Prandtl's analogy between the theory of torsion and that of a membrane, Wieghardt's analogy between the bending of a plate and the

stressing of a slice are discussed. Physical similitude and generalised physical similitude ("affinity") are discussed and exemplified. The analogy between the various phenomena governed by the Laplace equation in the plane, and their relation to the theory of functions of a complex variable is discussed. Finally, the analogies between the phenomena governed by the three-dimensional Laplace equation are pointed out and their presentation by regular quaternion functions which are independent of x_0 is explained on the basis of the investigations of Fueter. *P. Neményi.*

ALGEBRA

Salié, H. Über Abels Verallgemeinerung der binomischen Formel. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **98**, no. 4, 19–22 (1951).

The author derives an identity, too long for complete statement, which may be used to evaluate the sum

$$\sum_0^n \binom{n}{k} (x+a_k)^{k-\epsilon}(y+b_{n-k})^{n-k-\epsilon'}$$

with a_k and b_k arbitrary integers, ϵ and ϵ' zero or one independently. Abel's generalization of the binomial formula is the special case $a_k=ka$, $b_k=-(n-k)a$. *J. Riordan.*

Robinson, Robin. A new absolute geometric constant? Amer. Math. Monthly **58**, 462–469 (1951).

The author considers the following combinatorial problem. There are n straight lines in a plane, no three concurrent, and no two parallel; hence there are $\binom{n}{2}$ points of intersection. How many combinations n at a time of these points of intersection are such that no combination contains more than two collinear points? Correcting Whitworth [Choice and Chance . . . , 5th ed., George Bell, London, 1901] the author derives for the enumerating function g_n the recurrence relation $g_{n+1}=ng_n+\binom{n}{2}g_{n-1}$ and makes estimates of the number b in the asymptotic expression $2ng_n/n!\sim(bn)^{1/2}$ This is the "new absolute geometric constant" in question.

Reviewer's note: With the convention $g_0=1$, the exponential generating function for g_n is

$$\sum_0^\infty g_n t^n/n! = (1-t)^{-1/2}\exp\left(-\frac{t}{2}-\frac{t^2}{4}\right)$$

and expansion of the right hand side shows $b=(4/\pi)e^{-1/2}$ which agrees with the author's estimate to the six figures given. *J. Riordan* (New York, N. Y.).

Saxena, P. N. A simplified method of enumerating Latin squares by MacMahon's differential operators. II. The 7×7 Latin squares. J. Indian Soc. Agric. Statistics **3**, 24–79 (1951).

[For part I see the same J. **2**, 161–188 (1950); these Rev. **12**, 312.] By means of three theorems the author's enumeration method based on MacMahon's differential operators is further simplified. The method is then applied to a complete enumeration of the 7×7 squares. In accord with a previous result by A. Sade [Énumération des carrés latin . . . , Marseille, 1948; these Rev. **10**, 278], he finds 16,942,080 reduced 7×7 squares. A comparison between the author's methods and Sade's shows Sade's method to be considerably shorter and simpler. *H. B. Mann* (Columbus, Ohio).

***Wade, Thomas L. The Algebra of Vectors and Matrices.** Addison-Wesley Press, Inc., Cambridge, Mass., 1951. ix+189 pp. $4.50.

The preface states that this book is intended to give an elementary exposition of matrix algebra, "that exposition being articulated with the basic concepts of modern algebra, to wit, group, integral domain, field, ring, basis, dimension, and isomorphism". Further, the applied scientist is expected "to find here in a single source, in relatively simple, readable language, the essential features of matrix algebra". Such a formidable task puts the author in a dilemma, for on the one hand he must expound his arguments with the utmost simplicity illustrating his theorems in the first place by simple numerical examples, proving them first for the simplest cases and later for more general cases. On the other hand he must at the same time aim at a sophisticated point of view if ne is to "articulate these theorems with the basic concepts of modern algebra". The book may be said to conform to the author's intentions, but in the limited space at his disposal the execution of his plan allows the barest mention of many important topics and necessitates the omission of the proofs of some of the more fundamental theorems such as (p. 112) the fact that the minimum function of a matrix is a factor of its characteristic function, and theorem XIX, p. 170, on the reduction of a real symmetric matrix by a real orthogonal matrix to diagonal form. Hermitian and unitary matrices are not dealt with but a number of interesting topics such as the matrix representation of groups are introduced with little more than a definition. A pleasing feature of the book is the inclusion of interesting historical notes. There are many well chosen numerical illustrations as well as instructive exercises for the reader.

The contents include: Chap. 1, Groups, integral domains, fields, rings. Chap. 2, Vectors of two and three dimensions. Chap. 3, Linear dependence of vectors. Chap. 4, Vectors of n-dimensions, abstract vector spaces. Chap. 5, Elementary properties of matrices. Chap. 6, Symmetric and orthogonal matrices, inverses, solution of n linear non-homogeneous equations in n unknowns. Chap. 7, Groups of matrices, linear transformations, cogredient and contragredient vectors, similar matrices. Chap. 8, Cayley-Hamilton theorem, characteristic vectors, diagonalisation of symmetric matrices. Chap. 9, Rank, equivalent matrices, congruent matrices. Chap. 10, Bilinear and quadratic forms. Chap. 11, Applications of matrices to problems in mechanics, statistics and to multiple factorial analysis. There is also a useful bibliography. *D. E. Rutherford* (St. Andrews).

Drazin, M. P. On diagonable and normal matrices. Quart. J. Math. Oxford Ser. (2) **2**, 189–198 (1951).

An $n\times n$ matrix over an algebraically closed field is diagonable if it has a diagonal classical canonical form. The author gives a number of necessary and sufficient conditions that a matrix be diagonable. Over the complex field a matrix is normal if it commutes with its conjugate transpose. Conditions are also given that a matrix be normal. Most of the results are already known. *N. H. McCoy.*

Fage, M. K. On symmetrizable matrices. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 153–156 (1951). (Russian)

An $n\times n$ matrix $A=\|a_{ij}\|$ is defined to be "generally symmetrizable" if $a_{ij}=s_{ij}p_{ij}$ where $s_{ij}=s_{ji}$, $p_j\neq 0$ and "symmetrizable" if, moreover, the a_{ij} are real and the p_j are positive. Necessary and sufficient conditions for a matrix to be of either kind are given. These conditions generalize

a result stated by Kolmogoroff for transition probability matrices [Math. Ann. 112, 155–160 (1935)]. *M. Loève.*

Karpelevič, F. I. On the characteristic roots of matrices with nonnegative elements. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 361–383 (1951). (Russian)

The domain containing characteristic roots of $n \times n$ matrices with nonnegative elements and fixed maximum of moduli of the roots is found. This solves a problem stated by Kolmogoroff in connection with Markoff chains and partially solved by Dmitriev and Dynkin [same Izvestiya 10, 167–184 (1946); these Rev. 8, 129]. *M. Loève.*

Taussky, Olga. Classes of matrices and quadratic fields. Pacific J. Math. 1, 127–132 (1951).

It was previously shown by the author [Canadian J. Math. 1, 300–302 (1949); these Rev. 11, 3] that there is a one-to-one correspondence between matrix classes $S^{-1}AS$ and ideal classes in the ring generated by a root α of an irreducible algebraic equation $f(x)=0$ with rational integral coefficients, where A and S have rational integral elements and $\det(S)=\pm 1$. It is now proved that in case $f(x)$ is quadratic the transposed matrices correspond to the inverse ideal class. When the ideal class is of order two, examples in the field generated by $(410)^{\frac{1}{2}}$ show that the matrix class may or may not contain a symmetric matrix. *W. Givens.*

Markovitch, D. La méthode de E. Galois et la resolution des équations algébriques. Bull. Soc. Math. Phys. Serbie 2, nos. 3–4, 73–80 (1950). (Serbo-Croatian. French summary)

Expository paper.

Murnaghan, Francis D. A generalization of Hermite's law of reciprocity. Proc. Nat. Acad. Sci. U. S. A. 37, 439–441 (1951).

Expressed in terms of the plethysm of S-functions, Hermite's law of reciprocity is equivalent to the statement that the expansions $\{m\}\otimes\{n\}$ and $\{n\}\otimes\{m\}$ are identical for binary forms, i.e., for all partitions into not more than two parts. The author states a more general theorem, namely, that for binary forms

$$[\{m\}\otimes\{k\}]\{k-1\}=[\{m+1\}\otimes\{k-1\}]\{m\}.$$

By consecutively setting $k=m+1$, $m+2$, $m+3$, etc., Hermite's law can be deduced. Examples are given using the theorem to obtain the binary analysis of expansions of the form $\{n\}\otimes\{m\}$. The conjugate theorem is

$$[\{m\}\otimes\{1^k\}]\{k-1\}=[\{m+1\}\otimes\{1^{k-1}\}]\{m\}.$$

The theorems are used to obtain complete expansions of $\{m\}\otimes\{3\}$ for $m=3, 4, 8$. The theorems are not proved, but the author hopes to publish details in the Anais da Academia Brasileira de Ciências and to show how to determine $(n-1)$ term brackets in the analysis of $\{m\}\otimes\{n\}$.

D. E. Littlewood (Bangor).

Abstract Algebra

Hashimoto, Junji. On direct product decomposition of partially ordered sets. Ann. of Math. (2) 54, 315–318 (1951).

The author has proved previously [Math. Japonicae 1, 120–123 (1948); these Rev. 11, 5] that any two finite direct (cardinal) product decompositions of a connected (i.e., cardinal sum-indecomposable) partly ordered set have a common refinement. This result is now extended to infinite product decompositions. *G. Birkhoff.*

Jónsson, Bjarni. A Boolean algebra without proper automorphisms. Proc. Amer. Math. Soc. 2, 766–770 (1951).

An infinite Boolean algebra A is constructed, which has no proper automorphisms. Specifically, A is the Boolean algebra of all open-and-closed subsets of a suitable simply ordered set S in its intrinsic interval topology. (Since S is totally disconnected and compact in this topology, M. H. Stone's isomorphism between the automorphisms of A and those of S is applicable.) This construction solves problem 74 of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloq. Publ., v. 25, 2d ed., New York, 1948; these Rev. 10, 673]. *G. Birkhoff* (Cambridge, Mass.).

Finkbeiner, Daniel T. A general dependence relation for lattices. Proc. Amer. Math. Soc. 2, 756–759 (1951).

The author considers dependence relations in a general partly ordered set Q, generalizing some results of MacLane [Duke Math. J. 4, 455–468 (1938)] concerning dependence relations in unordered sets. For example, when Q satisfies the descending chain condition, he obtains sufficient conditions for the lattice $L(Q)$ of all closed subsets of Q to be semimodular, and for the set of completely irreducible elements of $L(Q)$ to be isomorphic with Q. Applications are planned to Dilworth's "arithmetical" theory of lattices [ibid. 8, 286–299 (1941); these Rev. 3, 100]. *G. Birkhoff.*

Ogasawara, Tôzirô, and Sasaki, Usa. On a theorem in lattice theory. J. Sci. Hiroshima Univ. Ser. A. 14, 13 (1949).

It is proved that any atomic lattice, each element of which has a unique complement, is a Boolean algebra. Previously, this had been proved (by a similar construction) under the extra hypothesis that L was complete [G. Birkhoff and M. Ward, Ann. of Math. (2) 40, 609–610 (1939); these Rev. 1, 2]. *G. Birkhoff* (Cambridge, Mass.).

Krishnan, V.-S. Les algèbres partiellement ordonnées et leurs extensions. Bull. Soc. Math. France 78, 235–263 (1950).

The author considers the completion in terms of the order of a partially ordered semi-group. The multiplication of the semi-group is order preserving, but is not defined in terms of the order. The author defines a "basic" extension by means of ideals but finds that this extension is neither minimal nor unique in any natural sense. A number of natural partial orders for the set of extensions of the given set are considered, their inter-relationships established, and their applications to the "basic" extension studied. The "basic" extension is applied to a number of common mathematical systems and examples illustrating the failure of the extension to be minimal or unique in the various senses considered are exhibited. This work generalizes that of the reviewer [Trans. Amer. Math. Soc. 42, 416–460 (1937)] where the multiplication is defined in terms of the order alone, and where the extension is minimal and unique in a natural sense. *H. M. MacNeille* (Providence, R. I.).

Devidé, Vladimir. Einige Beziehungen der Kommutativitäts- und der Assoziativitätseigenschaft. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 6, 33–48 (1951). (German. Serbo-Croatian summary)

Let S be a set of elements with a binary operation ω. It is assumed that S contains an identity element and at least

one element distinct from the identity. Let $\Pi_1(A, B, \cdots)$ and $\Pi_2(M, N, \cdots)$ be products satisfying the following conditions: 1) the number of factors in each is finite, 2) Π_1 and Π_2 are not formally identical, and 3) no variable appears more than once in each product. The operation ω is said to have the property (π) if $\Pi_1(A, B, \cdots) = \Pi_2(M, N, \cdots)$ for all $A, B, \cdots, M, N, \cdots$ in S. It is shown that the same variables must appear in Π_1 and Π_2, i.e. that ω has property (π) if the equation $\Pi_1(A, B, \cdots) = \Pi_2(A, B, \cdots)$ is satisfied for all $A, B, \cdots$ in S. If the property (π) of ω is not a consequence of associativity of ω, then (π) implies commutativity. If (π) is a consequence of associativity, then (π) implies associativity. These statements are valid when the terms associativity and commutativity are interchanged.

F. Kiokemeister (Chicago, Ill.).

Cohen, I. S., and Kaplansky, I. Rings for which every module is a direct sum of cyclic modules. Math. Z. **54**, 97–101 (1951).

It is known [Köthe, Math. Z. **39**, 31–44 (1934)] that if every module of a commutative ring R (with unit element) is a direct sum of cyclic modules and if R satisfies the descending chain condition then R is a principal ideal ring. In the present paper it is shown that the descending chain condition need not be assumed, but it follows automatically from the other assumptions. The proof starts with a consideration of orthogonal elements in R, which disposes of the semisimple case, and then uses the structural analysis of those commutative rings in which prime ideals are all maximal. *T. Nakayama* (Nagoya).

Gröbner, W. Ein Irreduzibilitätskriterium für Primärideale in kommutativen Ringen. Monatsh. Math. **55**, 138–145 (1951).

An irreducibility criterion for polynomial ideals, due to Macauley [Proc. Cambridge Philos. Soc. **30**, 27–46 (1934), p. 40, Th. VIII], is generalized to the following: Let I be an integral domain satisfying the maximal condition for ideals. Suppose, furthermore, that the following requirements are satisfied (as they are, for polynomial rings). (1) For any pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}'$, all ordered non-augmentable chains of prime ideals from $\mathfrak{p}$ to $\mathfrak{p}'$ have the same length. (2) The rank of a prime ideal $\mathfrak{p}$ is defined as the maximal length of the chains of prime ideals from (0) to $\mathfrak{p}$, and an ideal $\mathfrak{a}$, with a basis of r elements, is called r-principal ("Ideal der Hauptklasse r") if every prime ideal containing $\mathfrak{a}$ is of rank at least r. The second requirement is that every r-principal ideal be unmixed, i.e., that all its associated prime ideals be of the same rank, r.

A prime ideal $\mathfrak{p}$ of rank r is called regular if, in the corresponding local ring $I_{\mathfrak{p}}$, the extension ideal $\mathfrak{p}I_{\mathfrak{p}}$ has a basis of r elements (it is known that every basis must have at least r elements). In these terms, it is shown that if $\mathfrak{a}$ is an r-principal ideal and $\mathfrak{q}$ a primary component of $\mathfrak{a}$ whose prime ideal is regular, then $\mathfrak{q}$ is irreducible. The proof rests on the well known criterion (due to E. Noether and the author) that the primary ideal $\mathfrak{q}$ with prime ideal $\mathfrak{p}$ is irreducible if and only if $\mathfrak{q}:(\mathfrak{q}:\mathfrak{q}') = \mathfrak{q}'$, for every primary ideal $\mathfrak{q}' \supseteq \mathfrak{q}$ whose prime ideal is $\mathfrak{p}$. *G. Hochschild* (New Haven, Conn.).

√*van der Corput, J. G. **Le théorème fondamental de l'algèbre.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 11–18. Centre National de la Recherche Scientifique, Paris, 1950.

This paper contains a partial exposition of matters treated more fully in another publication [van der Corput, Math. Centrum Amsterdam, Scriptum no. 2 (1950); these Rev. **12**, 475]. *H. Levi* (New York, N. Y.).

√*Mahler, K. **On algebraic relations between two units of an algebraic field.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 47–55. Centre National de la Recherche Scientifique, Paris, 1950.

The principal result of this paper is this theorem: Let $\mathfrak{K}$ be a field of finite degree over the rational field, and let $F(x, y)$ be a polynomial with coefficients in $\mathfrak{K}$ which is irreducible over the field of all complex numbers. Assume that the curve C: $F(x, y) = 0$ is not a line parallel to either of the coordinate axes. (A) If there are infinitely many points (x, y) on C for which x is an integer and y a unit in $\mathfrak{K}$, then the curve may be expressed parametrically in the form $x = P(z) = \sum_{k=-m}^{+m} a_k z^k$, $y = Q(z) = \beta z^d$, where m and d are two positive integers, all coefficients a_k and $\beta \neq 0$ are in R, and where $P(z)$ is not a constant. (B) If there are infinitely many points (x, y) on C such that x and y are integers in $\mathfrak{K}$, the norm of y is unbounded, and y is divisible only by a finite set $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s\}$ of prime ideals, then the curve can be expressed parametrically in the form $x = \sum_{k=0}^{m} a_k z^k$, $y = \beta z^d$ where m and d are two positive integers, all coefficients a_k and $\beta \neq 0$ are in $\mathfrak{K}$, and $P(z)$ is not a constant.

This is an extension of a theorem of Siegel [Abh. Deutsch. Akad. Wiss. Math.-Nat. Kl. 1929, 1–70 (1930)] in which the additional assumptions about the integers y lead to additional information about the parametric representation. The author first shows that for each sufficiently large prime N there is an integer $\alpha \neq 0$ of K such that the curve C_N: $F(x, \alpha y^N) = 0$ also satisfies the hypotheses of Siegel's theorem, from which his result is quickly derived. He further extends the theorem by subjecting x as well as y to the additional hypotheses, obtaining the theorem: Let $\mathfrak{K}$ be a field of finite degree over the rational field, and let $F(x, y)$ be a polynomial with coefficients in $\mathfrak{K}$ which is irreducible over the complex field. Let the curve C: $F(x, y) = 0$ contain an infinite set of points (x, y) where x and y are units in $\mathfrak{K}$, or where, more generally, both x and y are divisible only by a finite set of given prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ in $\mathfrak{K}$. Then the polynomial $F(x, y)$ consists of exactly two non-vanishing terms. *H. Levi* (New York, N. Y.).

Amitsur, Shimshon. Finite differential polynomials. Riveon Lematematika **4**, 1–8 (1950). (Hebrew. English summary)

Let K be quasifield with a derivation $a \to a'$, let $K[t]$ be the ring of differential polynomials, i.e., the set of polynomials $\sum t^i a_i$, $a_i \in K$, with multiplication defined by $at = ta + a'$ for $a \in K$. A polynomial is said to finite if it is monic and the right ideal it generates is two-sided; this is equivalent to its being monic and contained in the center of $K[t]$. For characteristic zero, it was proved by Jacobson [Ann. of Math. (2) **38**, 484–507 (1937)] that there exist no finite polynomials except 1, unless the given derivation is inner, say $a' = ab - ba$. In this case the finite polynomials are the polynomials in $t - b$ over the center of K. The author assumes characteristic $p \neq 0$ and proves that there exists no finite polynomial except 1, unless the elements of K satisfy a differential equation of the form

$$z^{(p^n)} + z^{(p^{n-1})} a_1 + \cdots + z' a_n + zb - bz = 0,$$

where b and the a_i are constants (i.e., $b' = a_i' = 0$), and the a_i lie in the center of K. If this condition is satisfied and if e

is minimum, then the finite polynomials are the polynomials in

$$t^p+t^{p-1}a_1+\cdots+ta_n+b,$$

over the subfield of central constants of K. *I. S. Cohen.*

Igusa, Jun-ichi. Abelian extensions of function fields. Sūgaku (Mathematics) **1**, 13–18 (1947). (Japanese)

The content of this paper is published in German in J. Math. Soc. Japan **1**, 63–72 (1948); these Rev. **10**, 526.

K. Iwasawa (Princeton, N. J.).

Theory of Groups

***Sade, Albert. Quasigroupes.** Published by the author, Marseille, 1950. 16 pp.

In this paper a large number of results in the theory of quasigroups are announced. Proofs are not given, but a very complete bibliography is included, and a laudable effort has been made to correlate the work of previous writers on the subject. The results, which are too numerous to mention in detail, include a theory of singular divisors (a generalization of normal divisors) and considerable discussion of quasigroups which satisfy the law $(ab)b=a$. Several other special associative laws are considered, and also such special cases as idempotent, unipotent, and self-unit quasigroups.

D. C. Murdoch (Vancouver, B. C.).

Paige, L. J. Complete mappings of finite groups. Pacific J. Math. **1**, 111–116 (1951).

A biunique mapping θ of a group or quasigroup onto itself is called complete if the mapping $x\to x\theta(x)$ is biunique. The author gives several necessary and one sufficient condition for a group to possess a complete mapping. The relation between complete mappings and orthogonal Latin squares is also studied. *H. B. Mann* (Columbus, Ohio).

Douglas, Jesse. On the existence of a basis for every finite Abelian group. Proc. Nat. Acad. Sci. U. S. A. **37**, 359–362 (1951).

Douglas, Jesse. On the basis theorem for finite Abelian groups. II. Proc. Nat. Acad. Sci. U. S. A. **37**, 525–528 (1951).

Let G be a finite primary Abelian group. (I) If the elements of G are arranged in non-ascending order of their periods, and if systematically from left to right those elements are crossed out which are linear combinations of the elements not yet crossed out, then the remaining elements form a basis of G. (II) If in a set of generators of G the sum of the periods is a minimum, then they form a basis.

K. A. Hirsch (London).

van de Waerden, B. L. Example d'un groupe avec deux générateurs, contenant un sous-groupe commutatif sans système fini de générateurs. Nieuw Arch. Wiskunde (2) **23**, 190 (1951).

Correction (as noted in the review) to a paper of J. de Groot [same volume, 128–130 (1950); these Rev. **11**, 415].

R. C. Lyndon (Princeton, N. J.).

Rédei, L., und Szép, J. Über die endlichen nilpotenten Gruppen. Monatsh. Math. **55**, 200–205 (1951).

Let G be a finite nilpotent group, S a subgroup, and a an element of G of prime-power order p^s. Let $S(a)$ be the group generated by adjoining a to S, and $S(a^p)$ by adjoining a^p. The central result states that $S(a^p)=S(a)$ implies $S(a)=S$, with the analogous implication for the corresponding commutator groups. Several consequences are given, and examples to show that the condition of nilpotency cannot be dropped. *R. C. Lyndon* (Princeton, N. J.).

Suzuki, Michio. The lattice of subgroups of a finite group. Sūgaku (Mathematics) **2**, 189–200 (1950). (Japanese)

The contents of this paper are published in English in Trans. Amer. Math. Soc. **70**, 345–371, 372–386 (1951); these Rev. **12**, 586, 587. *K. Iwasawa* (Princeton, N. J.).

Petropavlovskaya, R. V. On the determination of a group by the structure of its subsystems. Mat. Sbornik N.S. **29(71)**, 63–78 (1951). (Russian)

A group G may be considered as an associative system (semigroup). If G contains elements of infinite order, then G will have subsystems which are not subgroups. The subsystems of an associative system will form a lattice, and systems A_1 and A_2 are said to be structure isomorphic if the corresponding lattices are isomorphic. The author endeavors to characterize groups in terms of their lattices of subsystems. It is shown that a system A, structure isomorphic to a group G containing an element of infinite order, will be a group G_1 and this isomorphism will make subgroups correspond. Also if a group G is abelian and contains an element of infinite order, a structure isomorphism between G and some other G_1 is in fact a group isomorphism. Similarly if a group G contains two independent elements of infinite order which permute, then a structure isomorphism between G and some other G_1 is in fact a group isomorphism.

Marshall Hall (Washington, D. C.).

Ėdel'man, S. L. On the p-normal series of a group. Doklady Akad. Nauk SSSR (N.S.) **79**, 209–212 (1951). (Russian)

The primary goal of this paper is to establish the following p-property extension of the Schreier theorem for groups: every two p-normal series of a group $\mathfrak{G}$ satisfying the δ-condition possess isomorphic refinements. A p-normal series of $\mathfrak{G}$ is a finite chain of subgroups from $\mathfrak{G}$ to the identity subgroup such that each term (after the first) is a proper p-normal subgroup of its predecessor. If, in a group $\mathfrak{G}$, $\mathfrak{G}^p$ denotes the subgroup generated by all elements of $\mathfrak{G}$ whose orders are non-negative powers of the prime p, then a subgroup $\mathfrak{N}$ of $\mathfrak{G}$ is a p-normal subgroup of $\mathfrak{G}$ provided $\mathfrak{N}$ permutes with each element of $\mathfrak{G}^p$. A group $\mathfrak{G}$ is said to satisfy the δ-condition provided for every two subgroups $\mathfrak{N}$ and $\mathfrak{H}$ such that $\mathfrak{N}\supset\mathfrak{H}$, the relation $\mathfrak{N}^p\cap\mathfrak{H}=\mathfrak{H}^p$ holds. A non-trivial illustration of a class of groups possessing this property is the class of p-decomposable groups [cf. Tchounikhin, C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 43–45 (1943); these Rev. **5**, 143]. *R. A. Good.*

Mal'cev, A. I. On some classes of infinite soluble groups. Mat. Sbornik N.S. **28(70)**, 567–588 (1951). (Russian)

This paper contains the full account, with proofs, of the results previously communicated [Doklady Akad. Nauk SSSR (N.S.) **67**, 23–25 (1949); these Rev. **11**, 78]. The only new theorem is Theorem 7: If every finite set of elements of a group G is contained in some soluble normal subgroup of it and if all Abelian subgroups of G have type 3 (i.e., the periodic part is direct product of a finite number of cyclic groups and groups of type p^∞, and the factor group has finite rank), then G is soluble. *K. A. Hirsch.*

Burger, E. Über Gruppen mit Verschlingungen. J. Reine Angew. Math. **188**, 193–200 (1950).

A linking (Verschlingung) of a finite abelian group $\mathfrak{G}$ is a symmetric bilinear function V which associates to every pair of elements g_1, g_2 of $\mathfrak{G}$ a rational number $V(g_1, g_2)$ (mod 1) in such a way that for every g_1 there is at least one g_2 for which $V(g_1, g_2) \not\equiv 0 \pmod 1$. A linking on a group is determined by the induced linkings on its primary components; the author therefore limits himself to groups of prime-power order p^n. The problem of finding a complete system of invariants for the linkings of a group was solved by Seifert [S.-B. Preuss. Akad. Wiss. 26–29, 811–828 (1933)] for odd primes p, and for the difficult case $p=2$ a solution was sketched by van Kampen [Amer. J. Math. 60, 595–610 (1938)]. The author reduces the problem to the Minkowski theory of quadratic forms by means of the following theorem: Denoting by $\mathfrak{B}=(v_{ik})$ the integral symmetric matrix $p^n(V(g_i, g_k))$, where the torsion numbers of G are $p^n \geqq p^n \geqq \cdots \geqq p^{n_s}$ and $g_1, \cdots, g_s$ is a basis of $\mathfrak{G}$, matrices $\mathfrak{B}_1$ and $\mathfrak{B}_2$ arise from isomorphic linkings (i.e., the one linking is carried into the other by an automorphism of $\mathfrak{G}$) if and only if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent mod p^n, i.e., there is an integral matrix $\mathfrak{A}$ with determinant $\not\equiv 0 \pmod p$ such that $\mathfrak{B}_2 \equiv \mathfrak{A}\mathfrak{B}_1\mathfrak{A}' \pmod{p^n}$. From this and a liberal use of the Minkowski theory he obtains as a complete system of invariants the number $\{m, p^n\}$ of mod p^n incongruent solutions of the congruence $\sum_{i,k=1}^{s} v_{ik}x_ix_k \equiv m \pmod{p^n}$, $m=1, 2, \cdots, p$. For odd p the Seifert invariants are derived from these. Finally the author solves the following two problems: Which matrices, and which classes of mod p^n equivalent matrices, arise from the linkings of a given group.

The reviewer wishes to note that the author's main result can be put into a very simple and direct form, valid for arbitrary finite abelian groups $\mathfrak{G}$: A complete system of invariants for a linking V of $\mathfrak{G}$ is $\{r\}$, r ranging over the rational numbers (mod 1), where $\{r\}$ denotes the number of elements g of $\mathfrak{G}$ such that $V(g, g) \equiv r \pmod 1$.

R. H. Fox (Princeton, N. J.).

Hurley, A. C. Finite rotation groups and crystal classes in four dimensions. Proc. Cambridge Philos. Soc. **47**, 650–661 (1951).

Goursat [Ann. Sci. École Norm. Sup. (3) **6**, 9–102 (1889)] enumerated the finite groups of congruent transformations in elliptic 3-space. Threlfall and Seifert [Math. Ann. **104**, 1–70 (1930)] made the larger enumeration of such groups in Euclidean 4-space. In the present paper, those groups are picked out which leave a lattice invariant. These are the four-dimensional analogues of the classical crystal classes. The author begins by computing the invariants (i.e., coefficients in the characteristic equation) for each element of each of Goursat's groups, and discarding any group in which non-integral invariants occur. For each of the remaining groups he considers the geometrical figure formed by the simplest invariant set of points in the manner of G. de B. Robinson [Proc. Cambridge Philos. Soc. **27**, 37–48 (1931)]. By seeing in which cases this figure forms part of a lattice, he shows that all but three of the groups with integral invariants have the desired property. He concludes that there are 222 finite crystal groups in four dimensions. Incidentally, he finds that Goursat made one mistake, omitting one of the possible ways of combining two dihedral groups. However, this omission had already been noticed by Threlfall and Seifert [loc. cit., pp. 14, 18, 22].

H. S. M. Coxeter (Toronto, Ont.).

Murnaghan, Francis D. The dimensions of the irreducible representations of a finite group. Proc. Nat. Acad. Sci. U. S. A. **37**, 441–442 (1951).

It is known that the dimension d of an irreducible representation of a finite group is a divisor of the order of the group, but the only known proof is not aesthetically satisfying. The author gives a new proof based on the multiplication law of the classes. If g is the order of the group and $d_1, \cdots, d_p$ the degrees of the representations, a matrix is obtained with integral elements and such that $g^2/d_1^2, \cdots, g^2/d_p^2$ are the latent roots. This implies that the latent roots are algebraic integers and hence rational integers.

D. E. Littlewood (Bangor).

Newell, M. J. On the representations of the orthogonal and symplectic groups. Proc. Roy. Irish Acad. Sect. A. **54**, 143–152 (1951).

The author reviews the known theory of the characters of irreducible representations of the orthogonal and symplectic groups and shows how unpublished results of his own can be applied to yield new proofs. *G. de B. Robinson.*

Newell, M. J. Modification rules for the orthogonal and symplectic groups. Proc. Roy. Irish Acad. Sect. A. **54**, 153–163 (1951).

The determinantal function $[\lambda]$ which furnishes the characters of the irreducible representations of the orthogonal group of $n=2k$ or $2k+1$ dimensions, provided that the number of partitions is not more than k, can still be evaluated if the number of partitions is $k+r$. Murnaghan [The theory of group representations, Johns Hopkins Press, Baltimore, 1938, p. 271] gave certain "modification rules" to cover the cases $r=1, 2$. The author proves that all functions $[\lambda_1, \cdots, \lambda_k, (\gamma)]$ vanish except for certain specified partitions (γ), and these non-vanishing functions are related to $[\lambda]$ in an explicit manner. He also studies the reduction of the Kronecker product of two representations which entails the analysis of the product $[\lambda]\cdot[\mu]$. The reduction is given in terms of certain associated Schur functions, and the results are applied to the study of concomitants under the orthogonal and symplectic groups. *G. de B. Robinson.*

Murnaghan, Francis D. The analysis of representations of the linear group. Anais Acad. Brasil. Ci. **23**, 1–19 (1951).

This is mainly an expository article describing some known results in the theory of matrix representations of the linear group. A matrix representation of the group of non-singular $n \times n$ matrices A is a $d \times d$ matrix $T(A)$ with elements polynomials in the elements of A, which satisfies $T(AB)=T(A)T(B)$. The spurs of such irreducible representations, or irreducible invariant matrices are called S-functions, being symmetric functions of the latent roots of A. The product of two S-functions corresponds to the Kronecker product of the representations. The evaluation of such products is discussed by the use of operators $\delta_1, \cdots, \delta_k$ where δ_i has the effect of increasing by 1 the ith of a sequence of k subscripts. (Similar operators have been used by M. J. Newell [see the preceding review].) Various specific products are evaluated. The problem of expressing an invariant matrix of an invariant matrix as a direct sum of invariant matrices is equivalent to the second type of multiplication of S-functions which is now called the plethysm of S-functions. This operation, denoted by $\{\lambda\} \otimes \{\mu\}$ is discussed and various exemplary evaluations are obtained. Use is made of $\{\lambda\} \otimes S_2$ after J. A. Todd [Proc. Cambridge Philos. Soc. **45**, 328–334 (1949); these Rev. **10**, 672].

D. E. Littlewood (Bangor).

Dieudonné, Jean. Sur les groupes orthogonaux rationnels à trois et quatre variables. C. R. Acad. Sci. Paris **233**, 541–543 (1951).

It is shown, in extension of earlier work of the author, that the orthogonal group in 3 variables over the rational field has a decreasing sequence $\{G_n\}$ of normal subgroups whose intersection is the identity element and such that G_n/G_{n+1} is finite, and abelian for sufficiently large n. The same is true for the pseudo-orthogonal groups in 4 variables when the ground form is indefinite or has discriminant equal to a square. *I. E. Segal* (Princeton, N. J.).

Thrall, R. M., and Robinson, G. de B. Supplement to a paper of G. de B. Robinson. Amer. J. Math. **73**, 721–724 (1951).

This note is a supplement to an earlier paper by Robinson [same J. **70**, 277–294 (1948); these Rev. **10**, 678]. A formula of the earlier paper, §7, p. 291, is corrected to give

$$\chi_\alpha(VW^{*+}) = \sigma_\alpha \cdot (\chi(W^*) \text{ in } [\alpha]_q^*)\chi_{\alpha^*}(V)$$

where W^{*+} is an element of cycle structure

$$(1^a q^{b_1}(2q)^{b_2}(3q)^{b_3}\cdots)$$

of the symmetric group of degree $n=a+bq$, W^* is of cycle structure $(1^{b_1}2^{b_2}3^{b_3}\cdots)$ of the symmetric group of degree b, V is a permutation of the a invariant symbols in W^{*+}, $[\alpha]_q^*$ denotes the representation corresponding to the star diagram of the representation α, $\sigma_\alpha=\pm 1$ according to the hook structure. The corrected formula is used to prove a new formula

$$\sum_{W^*\in S_b^*} \chi_\alpha(W^{*+})/b! = \theta_\alpha$$

where θ_α is zero if the q-core of $[\alpha]$ is not zero or if $[\alpha]$ has more than q rows, $\theta_\alpha=\sigma_\alpha$ otherwise. Some results are given associating the q-core of the representation $[\alpha_1, \alpha_2, \cdots, \alpha_k]$ with the set of residues (mod q) of the set of numbers α_i+m-i. *D. E. Littlewood* (Bangor).

Mautner, F. I. A generalization of the Frobenius reciprocity theorem. Proc. Nat. Acad. Sci. U. S. A. **37**, 431–435 (1951).

Let G be a locally compact unimodular group and K a compact subgroup of G, and let $K\to u(k)$ ($k\in K$) be a continuous unitary representation of K in a Hilbert space H_0. The set of all those "measurable" functions $X=X(g)$ defined on G and with values in H_0, which have finite norms $\|X\|^2=\int_G \|X(g)\|^2 dg<+\infty$ ($\|X(g)\|$ = the norm of $X(g)$ in H_0) and satisfy $X(kg)=u(k)X(g)$ for all $k\in K$ and $g\in G$, forms a Hilbert space H with the above norm $\|X\|$, and the operators $U(g)$ defined for any g in G by

$$(U(g)X)(g') = X(g'g), \quad g'\in G,$$

give a continuous unitary representation $g\to U(g)$ of G in H. The operator $U(g)$ is called the induced representation of G generated by the representation $u(k)$ of the subgroup K.

The author announces, without detailed proof, the following result. Let $H=\int H_t$ be the central decomposition of the above Hilbert space H with respect to the algebra of bounded operators in H generated by all $U(g)$, and let $g\to V_t(g)$ be the continuous unitary representation of G induced by $U(g)$ on each space H_t [cf. Mautner, Ann. of Math. (2) **52**, 528–556 (1950); these Rev. **12**, 157]. If $V_t(k)$ denotes the restriction of the above representation $V_t(g)$ of G to the subgroup K and W_t' the algebra of those bounded operators in H_t which commute with all $V_t(g)$ ($g\in G$), and if the representation $u(k)$ of K is irreducible, then the multiplicity of the representation $u(k)$ in $V_t(k)$ is equal to the dimension of W_t' over complex numbers for almost all t. This is a generalization of the well-known Frobenius reciprocity theorem on induced representations of compact (in particular, finite) groups to the case of locally compact G. *K. Iwasawa* (Princeton, N. J.).

Murakami, Shingo. On unitary representations of compact groups. Kōdai Math. Sem. Rep. **1951**, 15–18 (1951).

This paper gives new proofs of the following known results. Any cyclic unitary representation of a compact group is contained in the regular representation. Any irreducible unitary representation of a compact subgroup of a locally compact group G is extensible to an irreducible unitary representation of G. [See I. E. Segal, Bull. Amer. Math. Soc. **55**, 46 (1949), abstract 27t.] *D. Montgomery* (Princeton, N. J.).

Matsushita, Shin-ichi. On a kind of representation of topological groups. Mem. Fac. Sci. Kyūsyū Univ. A. **5**, 75–82 (1950).

The author studies completely regular topological groups through the unitary representations of the free topological group generated by them, and of which they are homomorphic images. Some of his proofs are unconvincing. *I. E. Segal* (Princeton, N. J.).

Gotô, Morio. Isomorphic representation of Lie groups. Sūgaku (Mathematics) **2**, 1–16 (1949). (Japanese)

The contents of this paper are published in English in Math. Japonicae **1**, 107–119 (1948); Nagoya Math. J. **1**, 91–107 (1950); these Rev. **10**, 681; **12**, 479. *K. Iwasawa* (Princeton, N. J.).

Kawada, Yukiyoshi. On the invariant differential equation of a Lie group. Sūgaku (Mathematics) **2**, 33–38 (1949). (Japanese)

The content of this paper is published in English in J. Math. Soc. Japan **1**, 217–225 (1949); these Rev. **12**, 77. *K. Iwasawa* (Princeton, N. J.).

Gleason, A. M. Compact subgroups. Proc. Nat. Acad. Sci. U. S. A. **37**, 622–623 (1951).

The author proves that every connected compact subgroup of a locally compact group is contained in a maximal connected compact subgroup. This strengthens the conjecture that all (connected) locally compact groups are generalized Lie groups [cf. Gleason, same Proc. **35**, 384–386 (1949); these Rev. **10**, 678]. The proof is based upon the following lemma: Let $\{H_\alpha\}$ be an increasing collection of connected closed subgroups of a locally compact group G, and let U be a neighborhood of e in G. Then there exists an index β such that all H_α are contained in $H_\beta U$. *K. Iwasawa* (Princeton, N. J.).

Vilenkin, N. Ya. On the existence of locally compact groups with given Ulm factors. Mat. Sbornik N.S. **29(71)**, 13–30 (1951). (Russian)

This is a continuation [see Mat. Sbornik N.S. **28(70)**, 503–536 (1951); these Rev. **13**, 110; and further references therein] of detailed proofs of results announced earlier [see Doklady Akad. Nauk SSSR (N.S.) **61**, 969–971 (1948); these Rev. **10**, 282]. The class of new objects studied in this sequel is the "pair" $(G:H)$ with G completely-regularly-stratified of type P, and H an open and closed regularly-stratified subgroup. Furthermore the set of elements of finite

order is assumed everywhere dense in G, and the subgroups G_α of the transfinite Ulm-sequence of subgroups of G are all of them required to be "servant-groups" of G. This means, as defined in earlier papers, that $p^\alpha G \cap G_\alpha = p^\alpha G_\alpha$, where for A in G, $p^\alpha A$ is the set of all $g = p^\alpha a$, $a \in A$. It is shown in Theorem 1 that the pair $(G:H)$ just described gives rise to the transfinite sequence of pairs $(G_\alpha:H_\alpha)$, each of which is a direct sum of finite cyclic groups, and a pair $(G_\tau:H_\tau)$ which is the direct sum of groups of type p^∞. The converse does not hold without a number of supplemental conditions. The third, and fundamental theorem shows that there always exists a pair $(G:H)$ as defined above, corresponding to any (reasonable) preassigned transfinite system of sums of finite cyclic groups, and a single final direct sum of groups of type p^∞. *L. Zippin* (Flushing, N. Y.).

Vilenkin, N. Ya. On the isomorphism of locally compact zero-dimensional Abelian groups with isomorphic factors. Mat. Sbornik N.S. **29(71)**, 31–62 (1951). (Russian)

It is proved that the pair $(G:H)$ determined in the paper reviewed above is unique, i.e., if $(G':H')$ is another pair corresponding to the same assigned invariants, then there exists an isomorphism of G onto G' such that H is mapped onto H'. The proof is given, first, for the case that G_τ and G'_τ are vacuous. The desired theorem then follows from theorem 2 that under the conditions on G there exists a closed subgroup T of G such that $G:H = T:(T \cap H) + G_\tau:(G_\tau \cap H)$. *L. Zippin* (Flushing, N. Y.).

Mostow, G. D. On an assertion of Weil. Ann. of Math. (2) **54**, 339–344 (1951).

The author studies the structure of Lie groups which have an invariant (by "invariant", we shall always mean "invariant under all inner automorphisms") compact neighbourhood of the identity. It is proved, among other results, that such a Lie group G is characterized by the following properties: (1) the topological commutator subgroup of the radical of G is compact, (2) the Lie algebra of G is decomposed into the direct sum of its semi-simple part and its radical, and (3) the group G modulo its radical is compact. The author also points out that it is untrue that the existence of an invariant compact neighbourhood of the identity always implies the existence of arbitrarily small invariant neighbourhoods of the identity as stated in A. Weil, L'intégration dans les groupes topologiques et ses applications, p. 129 [Actual. Sci. Ind., no. 869, Hermann, Paris, 1940; these Rev. 3, 198]. This inference holds, however, if G is simply-connected or if a neighbourhood of the identity of G is separated by finite-dimensional representations of G. In all the discussions, the Lie algebra of G plays an important rôle. *H. C. Wang* (Princeton, N. J.).

Iwasawa, Kenkichi. Topological groups with invariant compact neighborhoods of the identity. Ann. of Math. (2) **54**, 345–348 (1951). Please read the note on p. 1139

Using quite elegant methods, the author extends to general locally compact groups the results of Mostow [see the preceding review] concerning Lie groups possessing an invariant compact neighbourhood of the identity. It is proved that a connected, locally compact group contains an invariant compact neighbourhood of the identity if and only if its topological commutator subgroup is compact. The structure of such a group is explicitly given. As a consequence of these results and the theorem of Zippin [Proc. Amer. Math. Soc. 1, 309–315 (1950); these Rev. 12, 78] that a connected, locally compact, two-ended group G is homeomorphic with the topological product of a real line and a compact set, the author proves that such a group G is actually the direct product of a compact group and the additive group of reals. *H. C. Wang* (Princeton, N. J.).

Mibu, Yoshimichi. Generalization of the theory of Haar measure. Measures on uniform spaces which are invariant under a given group of homeomorphisms. Sūgaku (Mathematics) 1, 1–13 (1947). (Japanese)

Let E be a uniform space defined by a system of uniformities $\{V_\alpha\}$ and let E be locally compact and be the sum of a countable number of compact sets. Suppose that a transitive group of homeomorphisms G of E be given, such that $(p, q) \in V_\alpha$ implies $(\sigma(p), \sigma(q)) \in V_\alpha$ for every $\sigma \in G$ and V_α. Using a method similar to that for the case of Haar measures on locally compact groups, the author proves the existence of an invariant measure $m(B)$, derived from an outer measure $m^*(A)$ satisfying the conditions: 1) $m^*(\sigma A) = m^*(A)$ for all $\sigma \in G$; 2) every Baire set of E is measurable; 3) for any subset A, there exists a Baire set B with $m^*(A) = m(B)$; 4) if A is totally bounded, then $m^*(A) < \infty$.

The author defines a topology in G, so that it becomes a topological group, taking as neighborhoods of e the subsets $U(K, V_\alpha)$, consisting of all σ in G such that $(p, \sigma(p)) \in V_\alpha$ for every p in a compact subset K of E, and he shows that, if G is locally compact and is the sum of a countable number of compact sets with respect to that topology, the measure $m(B)$ has also other properties of Haar measures. It is then proved that G has the above mentioned topological structure and, consequently, $m(B)$ has nice properties, if G is the group of all isometric transformations of a metric space E and if either 1) E is compact or 2) E is connected or 3) every bounded closed subset of E is compact. Finally, the existence of non-measurable subsets with respect to an invariant measure on a group is also discussed. *K. Iwasawa.*

Maak, Wilhelm. Der Kronecker-Weylsche Gleichverteilungssatz für beliebige Matrizengruppen. Abh. Math. Sem. Univ. Hamburg **17**, 91–94 (1951).

The author states the Kronecker-Weyl approximation theorem in the following way. Let $\{D^1(x)\}$ and $\{D^2(x)\}$ be two classes of almost-periodic functions on a group, each being a ring and closed with respect to translations and conjugations. If their intersection consists only of constants, then for any continuous function $f(D^1(x), D^2(y))$ in the underlying matrices constituting the classes one has

$$M_x M_y f(D^1(x), D^2(y)) = M_x f(D^1(x), D^2(x)).$$

For Abelian groups this had been stated by Pontryagin. *S. Bochner* (Princeton, N. J.).

Gelbaum, B., Kalisch, G. K., and Olmsted, J. M. H. On the embedding of topological semigroups and integral domains. Proc. Amer. Math. Soc. **2**, 807–821 (1951).

Let S be a commutative semigroup with identity e in which the cancellation law holds and let R be the relation in $S \times S$ such that $(a, b) R (c, d)$ if and only if $ad = bc$. The quotient group $S \times S/R$ is denoted by $Q(S)$ and the canonical mapping from $S \times S$ onto $Q(S)$ by φ. For an integral domain I, R, φ and the quotient field $Q(I)$ are defined similarly. In the present paper, the authors consider topological semigroups and integral domains and study the relations between S, I and $Q(S)$, $Q(I)$ respectively in various topological aspects. Among others, the following theorems are proved: If the relation R is open and if $Q(S)$ $(Q(I))$ is endowed with the strongest topology with respect to which

φ is continuous, then $Q(S)$ ($Q(I)$) is a topological group (field). If $S(I)$ is a topological group (field), then it is identical with $Q(S)$ ($Q(I)$) topologized as above. If S is a compact semigroup, then it is a compact group. If S has an invariant metric, so does the group $Q(S)$ and if, moreover, R is open, then S is homeomorphically embedded in $Q(S)$. It is also proved that if S_1 and S_2 are complete separable metric semigroups, then, under some reasonable assumptions, every continuous isomorphism of S_1 onto S_2 is a homeomorphism. This is a generalization of a well-known theorem of Banach for complete separable metric groups.

K. Iwasawa (Princeton, N. J.).

Ritt, J. F. Subgroups of differential groups. Ann. of Math. (2) **54**, 110–146 (1951).

In this posthumous paper the author continues his study of differential groups [Ann. of Math. (2) **51**, 756–765 (1950); **52**, 708–726 (1950); **53**, 491–519 (1951); these Rev. **11**, 639; **12**, 241, 674]. In the preceding papers a differential group of order n was defined as n power series, with certain properties, in two sets of n indeterminates and their derivatives of various orders; in the present paper the author finds it expedient to introduce a space S_n for which these series yield a law of composition with respect to which S_n is a group, and then to regard S_n as the differential group. Starting with an ordinary differential field F of characteristic 0 (which is assumed to contain a nonconstant) and a constant h transcendental over F, let S_1 be the set of all power series $\sum_{j=1}^{\infty} a_j h^j$ with each $a_j \in F$; S_n is then defined as S_1^n. If $m \leqq n$ and $L_1, \cdots, L_m$ are homogeneous linear differential polynomials over F in $z_1, \cdots, z_m$ such that each z_i can be expressed linearly over F in terms of the L_j and their derivatives, the image s_m of a mapping $(z_i) \to (L_j + \cdots)$ of S_m into S_n (where $L_j + \cdots$ is a power series over F in the z_i and their derivatives) is called a subspace of S_n; if s_m is closed with respect to the law of composition then s_m is called a subgroup of order m. Results are proved analogous to Lie's first and second theorems for Lie groups. It is then shown that for two subgroups g_1 and g_2 of a differential group there exists a subgroup g contained in g_1 and g_2 which contains every subgroup contained in g_1 and g_2; g is called the intersection of g_1 and g_2, and need not coincide with the set-theoretic intersection. A subgroup g' called the product of g_1 and g_2 is defined in similar fashion. If a subgroup of order m is normal it is shown how to find a differential group of order $n-m$ isomorphic in a certain way to the usual group-theoretic factor group; such a differential group, which is unique up to equivalence, is called a factor group. It is shown finally that if g_1 and g_2 are normal of orders p and q then g and g' are normal and the sum of their orders is $p+q$.

E. R. Kolchin (New York, N. Y.).

NUMBER THEORY

✓***Nagell, Trygve. Introduction to Number Theory.** John Wiley & Sons, Inc., New York; Almqvist & Wiksell, Stockholm, 1951. 309 pp. $5.00.

This is a very readable introduction to number theory, with particular emphasis on diophantine equations, and requires only a school knowledge of mathematics. The exposition is admirably clear. More advanced or recent work is cited as background, where relevant. (Exceptionally, the Thue-Siegel theorem is quoted but not Dyson's improvement.) Necessarily, many of the proofs are fairly standard but there are some welcome novelties: Gauss's own evaluation of Gauss's sums (depending on a "basic" analogue of $\sum \binom{n}{r} = 2^n$), which is still perhaps the most elegant, is reproduced apparently for the first time. There are 180 examples, many of considerable interest, some of these being little known. The general scope of the book may be gathered from the following partial synopsis: Chapter I, Divisibility (unique factorisation, etc., also the irrationality of $\sqrt{2}$, e, π); Chapter II, Distribution of primes (sieve of Eratosthenes, $c_3 x < \pi(x) \log x < c_4 x$, estimation of $\sum p^{-1}$, $\sum p^{-1} \log p$, etc.); Chapter III, Congruences (Chinese remainder theorem, $f(x) \equiv O(p)$, Wilson's theorem, power residues, primitive roots, etc.); Chapter IV, Theory of quadratic residues; Chapter V, Arithmetical properties of roots of unity (irreducibility of the cyclotomic polynomial, its prime divisors, primes of the form $ny \pm 1$, Gauss's sums); Chapter VI, Diophantine equations of the second degree (existence of representations by certain binary quadratics using Thue's theorem that for given n, s there are integers x, y such that $xs \equiv y(n)$, $0 < \max(|x|, |y|) \leqq \sqrt{|n|}$, existence of representations by four squares, Pell's equation, integral and rational points on conics, $ax^2 + by^2 + cz^2 = 0$); Chapter VII, Diophantine equations of higher degree (infinite descent applied to $x^4 + y^4 = z^2$, etc., arithmetic in $K(\sqrt{-1})$, $K(\sqrt{-2})$ and $K(\sqrt{-3})$ and its application to diophantine equations, Mordell's finite basis theorem and Nagell's theorem on solution of finite order for curves of genus 1, application of the Thue-Siegel theorem (without proof of the theorem).) The final chapter (VIII) gives the "elementary" proof of the Prime Number Theorem (in Erdös's version), starting from scratch, and requires considerably more mental stamina than the rest of the book.

J. W. S. Cassels.

✓***Fueter, Rudolf. Synthetische Zahlentheorie.** 3rd ed. Walter de Gruyter & Co., Berlin, 1950. viii+248 pp. DM 20.

The 2d edition was published in 1925. In this edition the first two chapters (Zahlbereiche, der Körper der rationalen Zahlen) have been rewritten; the remaining six chapters have only minor changes.

Thébault, Victor. Sur les chiffres des carrés parfaits. Mathesis **60**, 182–184 (1951).

Webber, G. Cuthbert. Non-existence of odd perfect numbers of the form $3^{2\beta} \cdot p^{\alpha} \cdot s_1^{2\beta_1} s_2^{2\beta_2} s_3^{2\beta_3}$. Duke Math. J. **18**, 741–749 (1951).

Sylvester [C. R. Acad. Sci. Paris **106**, 522–526 (1888)] proved that no odd perfect number exists which has less than 5 different prime factors and stated without a proof that also no such number exists which has only 5 different prime factors. The author gives a proof of this theorem of Sylvester. While the paper was in the hands of the editor of the Duke Math. J., another proof of the same theorem was published by Kühnel [Math. Z. **52**, 202–211 (1949); these Rev. **11**, 714].

A. Brauer (Chapel Hill, N. C.).

de Bruijn, N. G., van Ebbenhorst Tengbergen, Ca., and Kruyswijk, D. On the set of divisors of a number. Nieuw Arch. Wiskunde (2) **23**, 191–193 (1951).

The degree $\delta(m)$ of a positive integer $m = p_1^{\lambda_1} \cdots p_r^{\lambda_r}$ is the sum $\lambda_1 + \cdots + \lambda_r$. A sequence $d_1, \cdots, d_h$ of divisors of m is

called a symmetrical chain if (a) $\delta(d_1)=\delta(m/d_h)$ and (b) d_{i+1}/d_i is a prime for $i=1,\cdots,h-1$. The authors prove that the set of all positive divisors of m can be decomposed into disjoint symmetrical chains. From this they deduce the following: (1) The set of all divisors d of m such that $\delta(d)=[\frac{1}{2}\delta(m)]$ contains as many members as any other set of divisors no one of which divides another. (2) Let S denote any set of divisors of m with the property that a positive integer d belongs to S if there are divisors d_1 and d_2 in S such that $d_1|d|d_2$. Then, $|\sum_{d\in S}\mu(d)|\leqq\binom{r}{[r/2]}$ where μ denotes the Möbius function. The theorems (1) and (2) are respectively generalizations of results of E. Sperner [Math. Z. 27, 544–548 (1928)] and P. Erdös [Math. Student 17, 32–33 (1950); these Rev. 11, 642]. *R. J. Levit.*

Brewer, B. W. Tests for primality. Duke Math. J. 18, 757–763 (1951).

The author uses the theory of Galois fields for the proofs of several tests for primality. Tests are obtained for 2^n-1 (the Lucas test: if n is an odd prime then $N=2^n-1$ is prime if and only if $N|S_{n-1}$, where $S_1=4$, $S_k=(S_{k-1})^2-2$); for $A2^n-1$ (A odd, $A<2^{n+1}$); also for $2p-1$, $4p-1$, $8p-1$, $2p^2-1$, $8p^2-1$ and $p^3\pm p-1$ for certain odd primes p. [There seem to be gaps in the proofs of theorems 7, 10 and 14; the possibility that n has prime factors $<p$ is overlooked.] *N. G. de Bruijn* (Delft).

Erdös, Pál. On a Diophantine equation. Mat. Lapok 1, 192–210 (1950). (Hungarian. Russian and English summaries)

For given integral a, b with $1\leqq a<b$ let $N(a,b)$ denote the smallest integer n such that

$$(1)\qquad a/b=\sum_{r=1}^{n}1/x_r,\quad\text{with}\quad x_1<x_2<\cdots<x_n,$$

is soluble in positive integers x_r. Sharpening a result and the method of N. G. de Bruijn he proves that

$$N(a,b)=O(\log b/\log\log b).$$

He conjectures that the true result is $N(a,b)=O(\log\log b)$, and he in fact establishes the inequalities

$$\sum_{a=1}^{b-3}N(a,b)>\tfrac{1}{2}(b-2)(\log\log b-1),\quad N(b-1,b)>\log\log b-1.$$

Other problems discussed include that of the greatest proper fraction a/b for which $N(a,b)\leqq n$, for given n, and that of the equation

$$(2)\qquad 1=\sum_{r=1}^{n}1/x_r,\quad\text{with}\quad x_1<x_2<\cdots<x_n,$$

and positive integral x_r. For (2) he conjectures that $x_n/x_1\geqq 3$, with equality only when $n=3$, $x_1=2$, $x_2=3$, $x_3=6$, and further that $\lim_{n\to\infty}x_n/x_1=\infty$. He also proves that (2) has only a finite number of solutions for given n. He also announces that he has proved that the "practical numbers" [Srinivasan, Current Sci. 17, 179–180 (1948); these Rev. 10, 356] have zero asymptotic density. *F. V. Atkinson.*

Obláth, Richard. Une équation diophantienne de M. Segre. Bull. Soc. Roy. Sci. Liège 20, 199–204 (1951).

A straightforward proof that $y^n=3x(x^3+x+1)$ ($n\geqq 2$) has no nontrivial integer solutions, together with trivial remarks on the possibility of rational solutions. Errera [same Bull. 19, 177–186, 213–214 (1950); these Rev. 12, 200] and Siegel [Errera, ibid. 19, 404–405 (1950); these Rev. 12, 853] have already shown that there are no nontrivial rational solutions for $n=2$. *J. W. S. Cassels* (Cambridge, England).

Roth, K. F. On the gaps between squarefree numbers. J. London Math. Soc. 26, 263–268 (1951).

Let q_n denote the nth square-free number. By means of a simple elementary argument it is shown that

$$q_{n+1}-q_n=O(x^{\frac{1}{2}});$$

a more elaborate argument leads to $q_{n+1}-q_n=O(x^{\frac{1}{3}+\epsilon})$. Finally, using a theorem of van der Corput [Math. Z. 28, 301–310 (1928), Th. 3], it is shown that

$$q_{n+1}-q_n=O(n^{3/13}(\log n)^{4/13}).$$

L. Carlitz (Durham, N. C.).

Halberstam, H., and Roth, K. F. On the gaps between consecutive k-free integers. J. London Math. Soc. 26, 268–273 (1951).

For $k\geqq 2$, let $q_k(n)$ denote the nth k-free integer (number not divisible by any kth power other than 1). It is proved that

$$q_k(n+1)-q_k(n)=O(n^{\frac{1}{2k}+\epsilon}).$$

(For $k=2$, see the preceding review.) It is stated that it is possible to improve this result and obtain an exponent less than $1/2k$. *L. Carlitz* (Durham, N. C.).

Turán, Paul. On the remainder-term of the prime-number formula. I. Acta Math. Acad. Sci. Hungar. 1, 48–63 (1950). (English. Russian summary)

Let $\Delta(x)=\psi(x)-x$ and $M(T)=\max_{1\leqq x\leqq T}|\Delta(x)|$. E. Schmidt [Math. Ann. 57, 195–204 (1903)] showed that $\Delta(T)=\Omega_{\pm}(T^{\frac{1}{2}})$. In this paper the author gives an explicit lower bound for $M(T)$. A specialization of his result is that there are positive numerical constants c and d (which may be calculated) such that for all $T>c$ we have

$$M(T)>T^{\frac{1}{2}}\exp(-d\log T\log\log\log T/\log\log T).$$

For integral x, $\Delta(x)=\sum_{n\leqq x}\{\Lambda(n)-1\}$. The author now defines $F(s)=\sum_{n=1}^{\infty}\{\Lambda(n)-1\}/n^s$ and

$$J=(2\pi i)^{-1}\int_{1+\eta-iU}^{1+\eta+iU}\xi^s s^{-k-1}F(s)\,ds,$$

where η, U, ξ, and k are suitable functions of T. The author easily obtains an upper estimate for J in terms of $M(T)$; but the determination of a lower estimate is more difficult and the author appeals to previous results of his [Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 197–262 (1947); J. London Math. Soc. 21, 268–275 (1947); these Rev. 9, 80] which give a lower bound for $|\sum_{n=1}^{s}z_n^r|$ for r in the interval $[m, m+N]$. These upper and lower estimates for J yield the inequality for $M(T)$. *L. Schoenfeld* (Urbana, Ill.).

Romanov, N. P. Hilbert space and the theory of numbers. II. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 131–152 (1951). (Russian)

In the first part of this paper [same Izvestiya Ser. Mat. 10, 3–34 (1946); these Rev. 8, 9] the author defined and considered the properties of D_g-sequences. The present paper contains applications of the results obtained there to different spaces of functions. Two methods are used. In the first any function $F(x)$ of $L^2(0,1)$ with the property

$$F(mx)=m^{s-1}\sum_{k=0}^{m-1}F\left(x+\frac{k}{m}\right)\quad\text{(all integers } m>0)$$

is taken, and from it is constructed a sequence of functions $f_n(x)=C^{-\frac{1}{2}}n^{\frac{1}{2}}F(nx-[nx])$ possessing the D_φ-property. Here $C=\int_0^1 F^2dx$. Orthonormal sequences can be constructed from such sequences $\{f_n\}$ by the methods described in the first part. In the second method a given orthonormal sequence $\{\alpha_n\}$ and multiplicative function $\omega(n)$ are taken, and from them a sequence possessing the D_φ-property is constructed (see review referred to above). This method has the advantage that if $\{\alpha_n\}$ is complete in a subspace H' of the space H so is the derived sequence. Non-trivial examples, which are too numerous and complicated to quote, are given for different choices of the space and involving, among other functions, the Bernouilli and Legendre polynomials, Laguerre functions as well as number-theoretical functions such as $\Lambda(n)$ and Dirichlet characters $\chi(n)$.

R. A. Rankin (Birmingham).

Vinogradov, I. M. An arithmetical method applied to questions of the distribution of numbers with a given property of the index. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 297–308 (1951). (Russian)

Let q be a prime, and n a divisor of $q-1$. The author defines $\sigma_r(x)$ to be $1-n^{-1}$ if the index of x (relative to a fixed primitive root mod q) is congruent to r (mod n), and to be $-n^{-1}$ if it is not. This defines $\sigma_r(x)$ for all x not divisible by q, and the definition is completed by putting $\sigma_r(x)=0$ for $x\equiv 0 \pmod q$. The function $\sigma_r(x)$ is periodic in x with period q, and periodic in r with period n, and the sum of its values over either period is 0. Theorem 1 of the present paper relates to the sum $S=\sum_{x=M+1}^{M+Q}\sigma_r(x)$, where $q^{\frac{1}{2}}<Q<\frac{1}{2}q$. If $h=Qq^{-\frac{1}{2}}$, it is proved that $|S|<q^{\frac{1}{2}}(a\log h+2.5)$, where $a=2(\log 27)^{-1}$. The proof is entirely elementary, and is almost the same as that already indicated by the author in the second half of a recent note [Doklady Akad. Nauk SSSR **73**, 635–638 (1950); these Rev. **12**, 161]. The only difference is that the present proof is expressed in terms of $\sigma_r(x)$ instead of in terms of Dirichlet characters. Theorem 2 relates to a double sum over both x and r in the case when $n=q-1$. Let $W=\sum_{x=M+1}^{M+Q}\sum_{r=N+1}^{N+Z}\sigma_r(x)$, where $q^{\frac{1}{2}}<Q<\frac{1}{2}q$, $n^{\frac{1}{2}}<Z<\frac{1}{2}n$. Then

$$|W|<q^{\frac{1}{2}}(a\log h+2.5)(a\log t+2.5),$$

where $h=Qq^{-\frac{1}{2}}$, $t=Zn^{-\frac{1}{2}}$. The proof is of a similar character to that of Theorem 1. Finally, Theorem 3 concerns the sum $\sum_{p\le n}\sigma_r(p+k)$, where p runs through primes, and $k\not\equiv 0 \pmod q$. If $N\ge 55$, it is proved that the sum is $O(N^{1+\epsilon}\{q^{-1}+qN^{-1}+N^{-1}\}^{\frac{1}{4}})$. This result is similar to one proved earlier by the author [same Izvestiya Ser. Mat. **7**, 17–34 (1943); these Rev. **5**, 143], but the proof now given is entirely elementary. The proof depends on Lemma 6 of the paper last quoted, and also on lemmas relating to sums of the form $\sum\sum\xi(x)\eta(y)\sigma_r(xy+k)$, where $\xi(x)$ and $\eta(y)$ are bounded functions, and summation is over the region defined by $U<x\le U'$ ($\le 2U$), $xy\le N$. *H. Davenport* (London).

Flett, T. M. On a coefficient problem of Littlewood and some trigonometrical sums. Quart. J. Math., Oxford Ser. (2) **2**, 26–52 (1951).

Let $-1<\Re(\tau_0)\le 1$, $\Im(\tau_0)>0$, $|\tau_0-\frac{1}{2}|\ge\frac{1}{2}$, $|\tau_0+\frac{1}{2}|>\frac{1}{2}$ and $\tau=(\tau_0+\tau_0\varepsilon)/(1+\varepsilon)$. In Whittaker and Watson's notation for the Jacobi theta-functions, let the modulus k, the function $\bar{Q}$ and the coefficients q_N be defined by

$$k(\tau)=\{\vartheta_2(0|\tau)/\vartheta_3(0|\tau)\}^2,$$

$$\bar{Q}(z)=2\log k(\tau)=\sum_{N=0}^{\infty}q_Nz^N, \quad |z|<1.$$

Letting $L_N=(\log N\log\log N)^{\frac{1}{2}}$, the author proves (carrying out the details only for the case $\tau_0=i$) that $|q_N|<B(\tau_0)L_N$. This is accomplished by working with the double series representation for q_N which in the case $\tau_0=i$ is

$$q_N=2i(-1)^N+32\pi^{-1}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}u_{n,m}\right),$$

$u_{n,m}=(-1)^{n-1}\{n^2+(2m-1)^2\}^{-1}\cos\{2N\arctan(2m-1)/n\}$. The main difficulty in the proof consists in estimating $|\sum_{n=1}^{n}u_{n,m}|$ for n in the interval $(2\exp(16L_N), N]$. For n in the interval $(N^{1/7}, N]$, this is accomplished by using results of van der Corput on the estimation of exponential sums. For n in the interval $[2\exp(16L_N), N^{1/7})$ the more difficult results of Vinogradov, here somewhat extended by the author, are used. By similar considerations, the author can show that Hecke's zeta function $\zeta(s;\lambda^k)$ in the field $k(i)$ satisfies the relation $\zeta(1;\lambda^k)=O(L_k)$ for large k. The various bounds for exponential sums are similar to those yielding the appraisal $\zeta(1+it)=O(L_t)$ which is substantially the best known result. *L. Schoenfeld* (Urbana, Ill.).

Fischer, Wilhelm. On Dedekind's function $\eta(\tau)$. Pacific J. Math. **1**, 83–95 (1951).

The author proves the transformation formula for Dedekind's function $\eta(\tau)=e^{2\pi i\tau}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})$, namely

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon\sqrt{\{-i(c\tau+d)\}}\eta(\tau),$$

where a, b, c and d are integers satisfying $ad-bc=1$, $c\ge 0$. The square root is the principal branch with positive real part and, since $\eta(\tau)$ is the 24th root of the discriminant, ϵ is a 24th root of unity which depends upon a, b, c and d in a rather complicated manner. Previous determinations of ϵ have used the theory of $\log\eta(\tau)$. The author evaluates ϵ directly from Euler's formula for $\eta(\tau)$ using only Poisson's formula for $\sum_{n=-\infty}^{\infty}\exp\{-\pi(m+\alpha)^2t\}$ ($\Re t>0$), and Gauss sums. *R. A. Rankin* (Birmingham).

Pennington, W. B. On the order of magnitude of Ramanujan's arithmetical function $\tau(n)$. Proc. Cambridge Philos. Soc. **47**, 668–678 (1951).

The best-known estimate for the order of magnitude of the "sum-function" $T(x)=\sum_{n\le x}\tau(n)$ of Ramanujan's $\tau(n)$ is $T(x)=O(x^{6-1/10})$, obtained by Rankin [Proc. Cambridge Philos. Soc. **36**, 150–151 (1940); these Rev. **1**, 203]. If, for prime p, the conjecture $|\tau(p)|\le 2p^{11/2}$ of Ramanujan were true, it would follow that $T(x)=O(x^{6-1/6+\epsilon})$. The author obtains results in the opposite direction, namely,

$$\liminf_{x\to\infty}\frac{T(x)}{x^{6-\frac{1}{2}}}=-\infty, \quad \limsup_{x\to\infty}\frac{T(x)}{x^{6-\frac{1}{2}}}=+\infty,$$

by a method similar to that developed by Ingham [ibid. **36**, 131–138 (1940); these Rev. **2**, 149] in treating classical lattice-point problems. Equation (12) should contain $\underline{\lim}$ instead of $\overline{\lim}$. *T. M. Apostol* (Pasadena, Calif.).

Lehmer, D. H. Ramanujan's function with respect to the modulus 49. Proc. Nat. Inst. Sci. India **17**, 67–70 (1951).

Let $\tau(n)$ be Ramanujan's function. If p is a prime which is a quadratic non-residue of 7 so that p is congruent to 3, 5 or 6 (mod 7), then $\tau(p)\equiv 3p(p^3+1) \pmod{49}$. This is proved by suitable manipulations of the Ramanujan identities connecting the power series with coefficient $1728\tau(n)$ and the power series whose coefficients are of the form $n^r\sigma_{s-r}(n)$. One consequence of this result is that if p and q are primes which

are quadratic non-residues of 7 and if 49 divides $p-q$, then 49 divides $\tau(p)-\tau(q)$. Another result is that if p is a prime, then 49 divides $\tau(p)$ if and only if p is congruent to 19, 31 or 48 (mod 49). *L. Schoenfeld* (Urbana, Ill.).

Maass, Hans. Über die Darstellung der Modulformen n-ten Grades durch Poincarésche Reihen. Math. Ann. **123**, 125–151 (1951).

This is an extension, to the Siegel matrix modular forms of degree n, of Petersson's theory of Poincaré series. A modular form of degree n and dimension $-k$ is an analytic function of the complex variables in the symmetric matrix (of degree n) $Z=(z_{\mu\nu})=X+iY$, regular in the half-space $Y>0$, and satisfying there the transformation equation (1) $g_{-k}(\sigma(Z))=|CZ+D|^k g_{-k}(Z)$ for every modular substitution $\sigma=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ of degree n. Siegel has shown that $g_{-k}(Z)$ possesses a Fourier expansion

$$(2)\qquad g_{-k}(Z)=\sum_{T\geqq 0} a(T)\exp 2\pi i\,\mathrm{Sp}\,(TZ),$$

extended over all non-negative symmetric matrices T with semi-integer elements; Sp = trace. Following Petersson ($n=1$), the author obtains from (2) by a process of summation a Poincaré series

$$(3)\qquad g_{-k}(Z,T)=\sum_{\sigma}\exp 2\pi i\,\mathrm{Sp}\,(T\sigma(Z))\times|CZ+D|^{-k}$$

with $T\geqq 0$, σ runs over a complete set of non-associated modular substitutions of degree n. (For the modular concepts, cf. Siegel, Math. Ann. **116**, 617–657 (1939); these Rev. **1**, 203.) The case $T=0$ yields the Eisenstein series.

The author's first goal is the proof of the following Theorem I. For $k>\min(2n, n+1+\text{rank } T)$, the Poincaré series (3) converges absolutely and uniformly in every closed bounded subdomain of $Y>0$ and defines there a modular form of dimension $-k$. The modular behavior of the Poincaré series would follow at once from its boundedness in $\mathfrak{F}_n$, the Siegel fundamental region of the modular group in the space $Y>0$. However, for $T\neq 0$, no majorant independent of Z is available. The author therefore expands the Poincaré series in a Fourier series

$$(4)\qquad g_{-k}(Z,T)=\sum_{T_0} a(T,T_0)\exp 2\pi i\,\mathrm{Sp}\,(T_0Z),$$

T_0 = symmetric semi-integral matrix, and shows that $a(T,T_0)=0$ whenever $T<0$, a condition equivalent to the boundedness of g_{-k}. This he does by introducing the scalar product (first defined by Petersson for $n=1$) of two modular forms

$$(5)\qquad (h(Z),g(Z))=\int\cdots\int_{\mathfrak{F}_n} h(Z)\overline{g(Z)}\,|Y|^{n-k-1}d(X)d(Y),$$

where $d(X)=\prod_{\mu\leqq\nu}dx_{\mu\nu}$, $dY=\prod_{\mu\leqq\nu}dy_{\mu\nu}$, $z_{\mu\nu}=x_{\mu\nu}+iy_{\mu\nu}$. He shows that the scalar product exists for $h=g_{-k}(Z)$, $g=g_{-k}(Z,T)$, provided that in the Fourier expansion (2) of $g_{-k}(Z)$ we have $a(T)=0$ whenever $|T|=0$—these are the so-called cusp-forms—and gives explicit formulas for the value of the scalar product; in particular, the scalar product is proportional to the Fourier coefficient $a(T)$.

The second object of the paper is the Representation Theorem II. Let $k\geqq 2n+2$ and even. Every modular form of degree n and dimension $-k$ can be written uniquely in the form $g_{-k}(Z)+h_{-k}(Z)$, $g_{-k}(Z)\in\mathfrak{R}_k^{(n)}$, $h_{-k}(Z)\in\mathfrak{S}_k^{(n)}$, where $\mathfrak{S}_k^{(n)}$ is the (linear) family of all cusp-forms of degree n and dimension $-k$, and $\mathfrak{R}_k^{(n)}$ is the family of all forms h_{-k} which are orthogonal to every cusp-form under the scalar-product (5). In the proof of this theorem important use is made of the following observation of Siegel. Write $Z=\left(\begin{smallmatrix}Z^* & 0\\ 0 & \lambda\end{smallmatrix}\right)$, where $Z^*=Z^{*(n-1)}$, n = column of zeros, $\lambda>0$. Then $g_{-k}(Z^*)=\lim_{\lambda\to\infty} g_{-k}(Z)$ is a modular form of degree $n-1$. Since the representation theorem for $n=1$ was proved by Petersson [Jber. Deutsch. Math. Verein. **49**, 49–75 (1939)], the author can proceed by induction.

The position of the Poincaré series in the family of modular forms is shown by Theorem III. Let $k\geqq 2n+2$ and even. The Poincaré series $g_{-k}(Z,T)$, where T has rank n, is orthogonal to every cusp-form whose Fourier coefficient $a(T)=0$. This property characterizes $g_{-k}(Z,T)$ uniquely apart from a constant factor. *J. Lehner.*

Szekeres, G. An asymptotic formula in the theory of partitions. Quart. J. Math., Oxford Ser. (2) **2**, 85–108 (1951).

This paper enriches our knowledge of the structure of the partitions of a positive integer by furnishing new results as well as by initiating a new method. Let $p(n,k)$ be the number of partitions of n into exactly k positive integral parts; $P(n,k)$, the number of partitions of n into k or fewer positive parts; $q(n,k)$, the number of partitions of n into exactly k positive unequal parts. An elementary result is (1) $P(n,k)\sim n^{k-1}/k!(k-1)!$ for k fixed, $n\to\infty$. Erdős and Lehner [Duke Math. J. **8**, 335–345 (1941); these Rev. **3**, 69; referred to as I] proved (1) for $k=o(n^{\frac{1}{2}})$, but could not extend it even to $k=O(n^{\frac{1}{2}})$. The author proves

$$(2)\qquad P(n,k)\sim[n^{k-1}/k!(k-1)!]\exp(k^2/4n-\cdots)\quad\text{for}\quad k=O(n^{\frac{1}{2}-\delta}),\quad \delta>0,$$

also $P(n,k)\sim p(n,k)$ for $k=o(n^{\frac{1}{2}})$; in I this result was conjectured for $k=O(n^{\frac{1}{2}})$ and proved for $k=o(n^{\frac{1}{2}})$.

It was shown in I that $k_0=2^{-1}cn^{\frac{1}{2}}\log n$, $c=6^{\frac{1}{2}}\pi^{-1}$, is the "normal" number of summands in a partition of n, i.e.,

$$(3)\qquad P(n,k)/p(n)=\{1+o(1)\}\exp(-ce^{-\lambda})\quad\text{for}\quad k=k_0+c\lambda n^{\frac{1}{2}},$$

where $p(n)=P(n,n)$ is the number of unrestricted partitions of n. Erdős [Bull. Amer. Math. Soc. **52**, 185–188 (1946); these Rev. **7**, 273] proved that $p(n,k)$, for fixed n, has a maximum at $k_1=k_0+c\log cn^{\frac{1}{2}}+o(n^{\frac{1}{2}})$, and that $q(n,k)$ has a maximum at $k_2=Acn^{\frac{1}{2}}+dn^{\frac{1}{2}}+o(n^{\frac{1}{2}})$ for a certain constant d; $A=2^{\frac{1}{2}}\log 2$. The author improves this to $k_2=Acn^{\frac{1}{2}}+B+O(n^{-\frac{1}{2}})$, with B a specified constant, so that $d=0$. Furthermore, for large n, k_2 is the nearest integer to $Acn^{\frac{1}{2}}+B$. By his method the author also obtains the first term of the Hardy-Ramanujan asymptotic expansions of $p(n)$, and of $q(n)$ (the number of partitions of n into unequal parts.)

In order to develop these results the author goes back to Sylvester's theory of "waves" [Amer. J. Math. **5**, 251–330 (1882)], which he combines with some complicated theorems of his own which yield asymptotic expansions for the coefficients of certain power series [Acta Sci. Math. Szeged **12B**, 187–198 (1950); these Rev. **13**, 220]. Observing that the residue of $(1-\varepsilon^{-t})^{-j}$ at $t=0$ is $+1$ for every integer $j\geqq 1$, Sylvester remarks that if $F(x)=N(x)/D(x)$ is a proper rational fraction, $F(x)=\sum_{\nu=-s}^{\infty}c_\nu x^\nu$, then $F(0)=\mathrm{Res}_{t=0}\sum_\nu F(a_\nu\varepsilon^{-t})$, where $D(a_\nu)=0$. The generating function of $P(n,k)$ being $\prod_{j=1}^{k}(1-x^j)^{-1}$, Sylvester sets

$$(4)\qquad P(n,k)=\sum_{q=1}^{k}\sum_{\zeta}P_\zeta(n,k)=P_1(n,k)+\sum_{q=2}^{k}\sum_{\zeta}P_\zeta(n,k),$$

where $P_\zeta(n,k)=\mathrm{Res}_{t=0}\,\zeta^{-n}e^{nt}\prod_{j=1}^{k}(1-\zeta^j e^{-jt})^{-1}$ and ζ runs over the primitive roots of unity of order $q\leqq k$. By using the above mentioned theorems, the author proves that

$P_1(n, k)$ is the principal term (corresponding in the Hardy-Ramanujan circle method to the "heaviest" singularity at $z=1$) and that it determines the asymptotic behavior of $P(n, k)$. The remaining sum, called the circulatory part, depends on the arithmetic structure of n.

The author points out by numerical examples the similarity of the Sylvester terms $P_1(n, n)$ to the terms of the Hardy-Ramanujan-Rademacher infinite series for $p(n)=P(n, n)$. Thus his method of proof opens the attractive possibility that the infinite series for $p(n)$ can be derived without using the theory of the (elliptic) modular functions. To the reviewer this is by far the most interesting aspect of the paper. *J. Lehner* (Philadelphia, Pa.).

✓ ***Châtelet, A. Utilisation des matrices dans l'algèbre et l'arithmétique des corps de nombres algébriques.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 21–25. Centre National de la Recherche Scientifique, Paris, 1950.

The author's objective is to show how matrix calculus can be employed to advantage in both the algebraic and arithmetic theories of algebraic numbers. On the algebraic side, he proves the well-known theorem that a separable algebraic extension field of finite degree over a field containing infinitely many elements is a simple extension, using matrix representations and avoiding the introduction of conjugate elements with the related normal extensions. On the arithmetic side, he shows how matrices can be employed to determine whether an ideal generated by two elements is principal or not. The former is believed by the author, and the reviewer, to be new. The latter is at least not "well-known". No references are given with the paper, but the author and others have applied matrix calculus to such problems before [cf. the author's book, Les groupes abéliens finis . . . , Gauthier-Villars, Paris, 1924; also C. C. MacDuffee's book, The Theory of Matrices, Ergebnisse der Math., Band 2, Heft 5, Springer, Berlin, 1933, especially Chapter III in which are several references to papers by Châtelet]. *R. Hull* (Lafayette, Ind.).

Brauer, Alfred. On the irreducibility of polynomials with large third coefficient. II. Amer. J. Math. **73**, 717–720 (1951).

The paper is a continuation of an earlier paper of the author [same J. **70**, 423–432 (1948), these Rev. **10**, 14]. The following main result is proved: Let

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n \quad (a_n\neq 0)$$

be a polynomial with integral rational coefficients. Let m and m^* be the minimum of the partial sums of the series $0+a_1+a_2+\cdots+a_n$ and $0-a_1+a_2-\cdots+(-1)^n a_n$, respectively. Assume that one of the following conditions is satisfied: $a_2\geqq\frac{1}{2}a_1^2+|m^*|$ for $n>2$ and $a_1>0$; $a_2\geqq\frac{1}{2}a_1^2+|m|$ for $n>2$ and $a_1<0$; $|a_1|\leqq 3$. The inequality $|a_v|\geqq$

$$\frac{1}{4}\binom{n-2}{v-2}\{|a_1|+n-2\}^2+\binom{n-2}{v-1}\{|a_1|+n-2\}+\binom{n-2}{v}$$

holds for at least one value of v ($v=2, 3, \cdots, n$). We set $t=|1+a_1|+|a_1+a_2|+|a_2|+|a_3|+\cdots+|a_n|$. If $a_2>t$, then $f(x)$ is irreducible in the field of rational numbers.

T. Nagell (Uppsala).

Beeger, N. G. W. H. On a new quadratic form for certain cyclotomic polynomial. Nieuw Arch. Wiskunde (2) **23**, 249–252 (1951).

Let $Q_n(x)$ be the usual cyclotomic polynomial whose roots are the primitive nth roots of unity. It is proved in this paper that, for n odd and squarefree and not divisible by 3, $Q_{12n}(x)=A_n(x)^2-(-1)^{\frac{1}{2}(n-1)}2nxB_n(x)^2$; here $A_n(x)$ and $B_n(x)$ are polynomials of degrees $2\varphi(n)$ and $2\varphi(n)-1$, respectively, whose coefficients are integers. The proof is based on the known identity $Q_{4n}(x)=R_n(x)^2-(-1)^{\frac{1}{2}(n-1)}2nxS_n(x)^2$, where n is odd and $R_n(x)$ and $S_n(x)$ are polynomials with integral coefficients; the relation $Q_{4n}(x)Q_{12n}(x)=Q_{4n}(x^3)$ plays an essential role in the proof. *H. W. Brinkmann.*

Faircloth, Olin B. A summary of new results concerning the solutions of equations in finite fields. Proc. Nat. Acad. Sci. U. S. A. **37**, 619–622 (1951).

The author discusses the solutions of an equation of the type $c_1x_1^{m_1}+c_2x_2^{m_2}+\cdots+c_nx_n^{m_n}+c_{n+1}=0$ where c_i and x_i are in a finite field $F(p^n)$, $c_1c_2\cdots c_n\neq 0$, and each m_i divides p^n-1. An expression for the number of solutions N of this equation is found in terms of the generalized Jacobi cyclotomic sums introduced by H. S. Vandiver [same Proc. **36**, 144–151 (1950); these Rev. **11**, 330; see also Faircloth and Vandiver, ibid. **36**, 260–267 (1950); these Rev. **12**, 81]. Various properties of N can be deduced from this; for example, it is easy to prove that $|N-p^{(s-1)n}|\leqq Ap^{\frac{1}{2}(s-1)n}+Bp^{\frac{1}{2}(s-2)n}$, where A, B are certain constants. It is also possible to obtain explicit expressions for N in certain special cases.

H. W. Brinkmann (Swarthmore, Pa.).

Heilbronn, H. On Euclid's algorithm in cyclic fields. Canadian J. Math. **3**, 257–268 (1951).

Let K be an absolutely cyclic field of degree k over the rational field R, and discriminant $\Delta=\Delta(K)$. Let $w(N)$ be the number of distinct rational primes which divide the rational integer $N\neq 0$. The author calls K of type T_1 if K is the composite of cyclic fields K_j, of degrees k_j relatively prime in pairs, and of discriminants Δ_j which are powers of different primes. He calls K of type T_2 if K is the composite of a K_1 of type T_1 and of odd degree, and a K_2 of degree 2^l, of discriminant Δ_2 such that $w(\Delta_2)\leqq k=\deg K$, and moreover such that when $l>1$ the unique subfield of K_2 of degree 2^{l-1} has a prime-power discriminant. The author considers class numbers H and h, in the narrow and wide sense, respectively, and proves that the g.c.d. $(k, H)>1$ if K is not of type T_1, and that $h>1$ if K is not of type T_2. These two theorems are obtained fairly directly from the class-field theory of which the author summarizes the results he needs in terms of character-chains. He also proves that there are only a finite number of K such that $k\geqq 4$, $w(\Delta(K))=1$, which are euclidean. This theorem is obtained, also using class-field theory, via a series of lemmas culminating in one which is a generalization of the "main lemma" of his earlier study of the case $k=3$ [Proc. Cambridge Philos. Soc. **46**, 377–382 (1950); these Rev. **11**, 716]. Finally, he combines his new theorems with those for $k=3$ and $k=2$ [ibid. **34**, 521–526 (1938)], and proves that there are only a finite number of cyclic fields of degree $k\geqq 2$, and belonging to any one of 11 types, which are euclidean. The 11 types are defined chiefly in terms of the above types T_1 and T_2, the values of $w(k)$ and $w(\Delta)$, and whether K is real or complex. The 11 types are not exhaustive. The author mentions two others which, as far as his studies go, may contain infinitely euclidean cyclic fields, for example, for $k=4$,

$R(\sqrt{\frac{1}{2}(5+\sqrt{5})}p)$, p a prime, $p\equiv 3 \pmod{20}$, $\Delta = 125p^2$. These quartics are also not included in Davenport's studies of the euclidean algorithm by quite different methods [cf. Trans. Amer. Math. Soc. 68, 508–532 (1950); these Rev. 12, 594]. *R. Hull* (Lafayette, Ind.).

Orsinger, Heinz. Zur Konstruktion von Trägheitsformen als Koeffizienten algebraischer Gleichungen. Math. Nachr. 5, 355–370 (1951).

Let y_ρ $(\rho=1, \cdots, r)$ be homogeneous forms of respective weights g_ρ of n variables x, with indeterminate coefficients a over a field k. The author extends results of Hurwitz, Perron and H. L. Schmid on forms and their related inertial forms. With skillful application of specializations $a \to a^* \epsilon k$ it is proved that, for $r > n$,

$$[K(x_1, \cdots, x_n) : K(y_1, \cdots, y_r)] = d = (g_1, \cdots, g_r),$$

$$x_\sigma^d p_0(y) + p_d(y) = 0$$

with $p_0,\ p_d \epsilon K[y_1, \cdots, y_n]$ and that the x_σ are primitive elements, where K is the field obtained from k by adjunction of the indeterminate coefficients of the forms y_ρ. This result requires that k have the characteristic 0 since the technique of proof uses the structure of the inertial fields of the specializations $a \to a^*$. The above statements remain valid though for prime characteristic provided certain (sufficient) conditions which are implied by the technique of proof for the case of characteristic 0, are made. Furthermore it is shown that the coefficients of the generating equations

$$p_0^{(\nu)}(y_{\rho_1}, \cdots, y_{\rho_s}) x_\sigma^\delta + p_\delta(y_{\rho_1}, \cdots, y_{\rho_s}) = 0$$

(made to be of homogeneous weight in the coefficients), $n < s \leqq r$, δ the g.c.d. of the degrees of the forms y_{ρ_i}, are inertial forms; now the x_σ are considered as primitive elements of $K(x_1, \cdots, x_n)/K(y_{\rho_1}, \cdots, y_{\rho_s})$. *O. F. G. Schilling.*

Dénes, Peter. Über Einheiten von algebraischen Zahlkörpern. Monatsh. Math. 55, 161–163 (1951).

Let K be an algebraic number field, and let k be a subfield of K. By the Dirichlet unit theorem, the group of units U_K of K is the direct product of the group V_K of the roots of unity in K by a free abelian group of rank $R = R_1 + R_2 - 1$, where R_1 is the number of real conjugates of K and $2R_2$ is the number of complex conjugates of K. Let r, r_1, r_2 denote the corresponding numbers for k. The author considers the situation where $R = r$. In that case, if $U_k = V_k \times F_k$, with F_k free abelian, the factor group U_K/F_k is evidently of finite exponent. It follows immediately that if σ is an isomorphism of K onto one of its conjugates which leaves the elements of k fixed, and $u \epsilon U_K$, then $\sigma(u)u^{-1}$ is a root of unity. The assumption $R = r$ holds in particular when K has a real subfield k over which it is of degree 2, for then $r_2 = 0$, $r_1 = R_2$, and $R_1 = 0$. Using the above, the author concludes that then the square of every unit of K is the product of a root of unity in K by a real unit of K. If all the roots of unity in K are of odd order it follows that every unit of K is the product of a root of unity in K by a unit which is either real or purely imaginary. *G. Hochschild* (New Haven, Conn.).

Ankeny, N. C., Artin, E., and Chowla, S. The class-number of real quadratic fields. Proc. Nat. Acad. Sci. U. S. A. 37, 524–525 (1951).

The following results are announced: Let $F = R(\sqrt{d})$ be the real quadratic field of fundamental discriminant d. Let h be the class number of F and $\epsilon = \frac{1}{2}(t + u\sqrt{d}) > 1$ be its fundamental unit. Let $d = pm$, where p is an odd prime and m an integer. If $p > 3$ and $m > 1$, then

$$-2hu/t \equiv \sum_{0<\nu<d} \frac{\chi(\nu)}{m\nu} \left[\frac{\nu}{p}\right] \pmod{p},$$

where $[x]$ denotes the greatest integer contained in x and $\chi(\nu)$ is the character belonging to F. If $p = 3$, then there is an additional factor $m+1$ on the left side of this equation. If $d = p \equiv 1 \pmod 4$, then $2hu/t \equiv (A+B)/p \pmod p$, where A is the product of all quadratic residues of p lying between 0 and p, and B is the product of all quadratic non-residues in the same interval. *W. H. Mills.*

Linnik, J. V. Quaternions and Cayley numbers. Math. Centrum Amsterdam. Rapport ZW-1951-002, 19 pp. (1951). (Dutch)

Translation of the introduction (pp. 49–65) of a paper in Uspehi Matem. Mauk (N.S.) 4, no. 5(33), 49–98 (1949); these Rev. 11, 163.

Cohn, Harvey. On finiteness conditions for a convex body. Proc. Amer. Math. Soc. 2, 544–546 (1951).

The following theorem is proved. Let K be a convex body in d-dimensional Euclidean space containing as its interior the hypersphere $\Sigma_d(r)$ of radius r and centre at the origin. Then if K fails to contain in its interior those lattice points (other than the origin) in a second sphere $\Sigma_d(C_d r^{1-d})$ it will contain in its interior no lattice points at all except the origin. In fact K will then lie entirely within the second sphere. This is proved by applying Minkowski's theorem to the body formed by joining the two points at the ends of a diameter of $\Sigma_d(\rho)$ to a circle of radius r centred at the origin and lying in a plane perpendicular to the diameter. This method gives $C_d = 2^{d-1} d \pi^{-\frac{1}{2}(d-1)} \Gamma\{\frac{1}{2}(d+1)\}$. It is shown how this value of C_d may be improved slightly, and it is proved by means of Perron's transferal principle [J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 67] that numbers C_d for which the theorem is true possess a positive lower bound C_d^* which is such that, as $d \to \infty$, $\log C_d^*/(d \log d)$ lies between $\frac{1}{2}$ and -1. *R. A. Rankin.*

Cohn, Harvey. On the finite determination of critical lattices. Proc. Amer. Math. Soc. 2, 547–549 (1951).

Let K be a finite convex body symmetric about the origin. The critical lattices for K are determined by minimising a determinant subject to an infinity of inequalities. Minkowski's finite basis theorem is used, in conjunction with the theorem proved in the paper reviewed above, to show that this infinite set of inequalities can be reduced to a finite set whose number depends only on the dimension of the space. *R. A. Rankin* (Birmingham).

Mori, Shinjiro. Über ganzzahlige quadratische Gleichungen, die eine reduzierte Zahl als Wurzel besitzen. J. Sci. Hiroshima Univ. Ser. A. 14, 8–12 (1949).

The author indicates an elementary method of proof for Lagrange's theorem on the periodicity of the simple continued fraction expansion for real quadratic irrationals. He gives a definition equivalent to that of Gauss for a reduced quadratic irrational x_0, namely, that x_0 be a positive root of an irreducible quadratic equation $ax^2 - bx - c = 0$, where a,

b, c are positive integers, and $b>|a-c|$. This definition is used to give direct proofs of well known results on purely periodic simple continued fractions and Pell's equation.

W. T. Scott (Evanston, Ill.).

Pipping, Nils. Eigenschaften der Diagonalkettenbrüche. Acta Acad. Aboensis **17**, no. 3, 23 pp. (1951).

The author continues investigations of two earlier papers [same Acta **15**, no. 10 (1947); **16**, no. 5 (1949); these Rev. **8**, 501; **10**, 509]. Conditions are given in terms of the elements of the simple continued fraction (RK) for $D^{\frac{1}{2}}$ (D a non-square positive integer) for determining those partial denominators a_n of the diagonal continued fraction (DK) for $D^{\frac{1}{2}}$ for which $a_n \geqq b_0$, where b_0 is the initial element of the RK. When the DK and RK for $(b_0^2+m)^{\frac{1}{2}}$ are identical it is shown under suitable conditions that the DK for $(b_0^2-m)^{\frac{1}{2}}$ is obtained from that for (b_0^2+m) by changing the partial numerators from $+1$ to -1. Tables of $D<10{,}000$ are given for which DK and RK expansions are identical.

W. T. Scott (Evanston, Ill.).

Mahler, K. On the generating function of the integers with a missing digit. J. Indian Math. Soc. (N.S.) Part A. **15**, 33–40 (1951).

Translated from a paper which appeared in Chinese in K'o Hsüeh (Science) **29**, 265–267 (1947); these Rev. **9**, 79.

Fel'dman, N. I. On the joint approximation by algebraic numbers of the logarithms of several algebraic numbers. Doklady Akad. Nauk SSSR (N.S.) **75**, 777–778 (1950). (Russian)

This note announces a generalization to the simultaneous approximation of the logarithms of several algebraic numbers by algebraic numbers of the results in his earlier paper [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 53–74 (1951); these Rev. **12**, 595]. *J. W. S. Cassels.*

Šapiro-Pyateckiĭ, I. I. On the laws of distribution of the fractional parts of an exponential function. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 47–52 (1951). (Russian)

Let q be an integer greater than unity and α a real number. The fractional parts $\{\alpha q^k\}$ for $k=0, 1, 2, \cdots$ determine a distribution function $\sigma(x)$. Conversely, if a distribution function $\sigma(x)$, is given it may be possible to find an α such that $\sigma(x)$ is the distribution function for the sequence $\{\alpha q^k\}$. It is proved that a necessary and sufficient condition for a distribution function $\sigma(x)$ to be that of a sequence $\{\alpha q^k\}$ $(k=0, 1, 2, \cdots)$ is that, for every continuous function $f(x)$ of period 1,

$$\int_0^1 f(x)d\sigma = \int_0^1 f(qx)d\sigma.$$

The necessity of the condition is easily shown. To prove it sufficient use is made of the theory of dynamical systems and work of Krylov and Bogolyubov. A criterion for uniform distribution is also given. *R. A. Rankin.*

Korobov, N. M. Normal periodic systems and their applications to the estimation of sums of fractional parts. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 17–46 (1951). (Russian)

Throughout q denotes an integer greater than unity. There are two parts to the paper. In part I the author defines a normal periodic system $\rho_n(q)$ to be a sequence of numbers $\delta_1\delta_2\cdots\delta_t$, where $t=q^n+n-1$, δ_i is an integer in the range $0\leqq\delta_i\leqq q-1$ $(i=1, 2, \cdots, t)$ and the sequence has the property that no set of n successive numbers occurs more than once. The number of sets of n successive numbers in the sequence is q^n and is the maximum number of different sets possible. The existence of such systems was proved by Good [J. London Math. Soc. **21**, 167–169 (1946); these Rev. **8**, 430], and the author develops two general methods for constructing them. His first method (method A_1) is simple, particularly for $q=2$, but does not give all normal periodic systems. For in the case $q=2$ method A_1 gives not more than 2^n systems, while de Bruijn has shown that the total number is 2^r, where $r=2^{n-1}-n$ [Nederl. Akad. Wetensch., Proc. **49**, 758–764 = Indagationes Math. **8**, 461–467 (1946); these Rev. **8**, 247]. By a very complicated analysis involving the consideration of certain special systems of digits associated with a given $\rho_n(q)$, the author derives a second method (method A_2), by means of which every normal periodic system can be obtained. Every system $\rho_n(q)$ has the property that its last $n-1$ digits are the same as the first $n-1$, and by omission of these final $n-1$ digits we obtain an associated system $\rho'_n(q)$.

In part II these systems are applied to the problem of the behaviour of the sum of the fractional parts of the function q^x $(x=0, 1, 2, \cdots)$, where x is a real number. If these fractional parts are uniformly distributed in the interval $(0, 1)$, write $\alpha\varepsilon L(q)$. Also write $E(\mu, P)=\sum_{x=1}^{P}\{\alpha q^x\}-\frac{1}{2}P$, where $\mu=1, 2, 3, \cdots$, so that, if $\alpha\varepsilon L(q)$ then $E(1, P)=o(P)$.

The following theorems are proved: I. Let $\epsilon(P)$ be any positive function which tends to zero as $P\to\infty$; then there exists an $\alpha\varepsilon L(q)$ such that $E(1, P)=\Omega(P\epsilon(P))$. II. Let $\varphi(P)$ be any positive function which tends to infinity as $P\to\infty$; then there exists an $\alpha\varepsilon L(q)$ such that $E(1, P)=O(\varphi(P))$, and for every $\alpha\varepsilon L(q)$ it is not possible to replace this estimate by $O(1)$. The author has shown that $\alpha\varepsilon L(q)$ implies $\alpha\varepsilon L(q^\mu)$ for $\mu=1, 2, 3, \cdots$ [Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 215–238 (1950); these Rev. **12**, 321]. That a similar result does not in general hold for $E(\mu, P)$ is shown by III. Let $\epsilon(P)$ and $\varphi(P)$ be as in I and II. Then there exists an $\alpha\varepsilon L(q)$ such that $E(1, P)=o(\varphi(P))$, $E(2, P)=\Omega(P\epsilon(P))$. IV. On the other hand, for any $\varphi(P)$ as in II, there exists an $\alpha\varepsilon L(q)$ such that $E(\mu, P)=o(P)$ for $\mu=1, 2, 3, \cdots$. V. Also, if $E(1, P)$ is sometimes large, $E(2, P)$ can be small for all P. Thus, for any $\epsilon(P)$ and $\varphi(P)$ as in I and II, we can find an $\alpha\varepsilon L(q)$ such that $E(1, P)=\Omega(P\epsilon(P))$, $E(2, P)=o(\varphi(P))$. VI. On the other hand there exist an $\alpha\varepsilon L(q)$ such that $E(\mu, P)=\Omega(P\epsilon(P))$ for each $\mu=1, 2, 3, \cdots$. These results are compared with the known results for the fractional parts of αx $(x=1, 2, 3, \cdots)$. Thus for αx I is true, but II is not true if $\varphi(P)$ increases slower than $\log P$. On the other hand $\sum_{x=1}^{P}\{\alpha x\}-\frac{1}{2}P=O(\log P)$ for almost all α while Khintchine has shown that $E(1, P)=O(\{P\log\log P\}^{\frac{1}{2}})$ for almost all α [Fund. Math. **6**, 9–20 (1924)].

In the proofs of these results α is expressed as a "decimal" to the base q in the form $0\cdot\delta_1\delta_2\delta_3\cdots$ and, where necessary, the sequence of δ_i is chosen from the digits of systems $\rho_n'(q)$, the rate at which n increases being determined by the given functions $\varphi(P)$ and $\epsilon(P)$. *R. A. Rankin*

ANALYSIS

Bojanić, R. On the first mean-value theorem. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 187–196 (1950). (Serbo-Croatian)

The author improves a result of R. Rothe [Math. Z. **9**, 300–325 (1921)] concerning the mean value theorem

$$(*) \qquad F(y)-F(x)=(y-x)F'\{f(x,y)\}.$$

It is shown that, for a given $f(x, y)$ defined in a square and having continuous partial derivatives $f_x=p$, $f_y=q$, $f_{xy}=s$, the functional equation (*) admits a solution $F(x)$ if and only if $f(x, y)$ is symmetric with $x\leqq f(x, y)\leqq y$ for $x\leqq y$, and $s/pq+(p-q)/(y-x)pq$, $y\neq x$, is a function only of f. Examples are given; cf. R. Bojanić [Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 105–111 (1949); these Rev. **11**, 716].

E. F. Beckenbach (Princeton, N. J.).

Sapogov, N. A. On an inequality of Čebyšev. Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 157–159 (1951). (Russian)

The inequality of Čebyšev,

$$(*) \qquad \int_a^b u(x)v(x)dx\geqq\frac{1}{b-a}\int_a^b u(x)dx\int_a^b v(t)dt,$$

holds for nondecreasing real functions $u(x)$, $v(x)$; it has been generalized on various occasions (for example by F. Franklin [Amer. J. Math. **7**, 377–379 (1885)]), as noted by the author. There are now given still wider generalizations, as suggested by the following simple proof: There is clearly a value c, $a\leqq c\leqq b$, such that the function

$$Q(x)=v(x)-\frac{1}{b-a}\int_a^b v(t)dt$$

satisfies $Q(x)\leqq 0$, $a\leqq x<c$; $Q(x)\geqq 0$, $c<x\leqq b$; hence

$$\int_a^b u(x)\left[v(x)-\frac{1}{b-a}\int_a^b v(t)dt\right]dx\geqq u(c)\int_a^b Q(x)dx=0,$$

from which (*) follows. It is shown, for instance, that for integrable real functions u, v, w defined on a set E, if e denotes the part of E for which

$$v(x)\int_E w(x)d\mu-w(x)\int_E v(x)d\mu>0,$$

and if $u(x')\geqq u(x'')$ for all x' in e and all x'' in E, then

$$(**) \qquad \int_E w(x)d\mu\int_E u(x)v(x)d\mu>\int_E v(x)d\mu\int_E u(x)w(x)d\mu.$$

A more immediate generalization of (*) is the special case $w\equiv 1$. Again, an inequality of Jensen [Bull. Sci. Math. (2) **12**, 134–135 (1888)],

$$\sum u_iw_i/\sum v_i\geqq\sum u_iw_i/\sum w_i$$

for $3n$ positive quantities satisfying $u_1\geqq u_2\geqq\cdots$ and $v_1/w_1\geqq v_2/w_2\geqq\cdots$, is contained in (**).

E. F. Beckenbach (Princeton, N. J.).

Viglino, Giacomo. Sul teorema di Weierstrass per la rappresentazione delle funzioni continue mediante serie di polinomi. Rivista Mat. Univ. Parma **2**, 103–109 (1951).

The author gives a simple and elementary proof of the Weierstrass approximation theorem. The given continuous function is first approximated by rational functions and then by polynomials. *A. Rosenthal* (Lafayette, Ind.).

Mikusiński, Jan G.-. Remarks on the moment problem and a theorem of Picone. Colloquium Math. **2**, 138–141 (1951).

On démontre le théorème suivant: Si la fonction $f(x)$ est intégrable dans l'intervalle fini $0\leqq a<x<b$, et si, designant par $\alpha_n=\alpha+n\beta$ une progression arithmétique de termes positifs, on a, pour un $\epsilon>0$,

$$\int_a^b x^{\alpha_n}f(x)dx=o[(a+\epsilon)^{\alpha_n}],$$

alors $f(x)=0$ presque-partout dans (a, b). L'introduction du terme ϵ doit servir évidemment pour comprendre le cas de $a=0$. D'autre côté l'hypothèse équivaut évidemment à supposer qu'il existe un q $(0<q<1)$ tel que

$$\lim_{n\to\infty}q^n\int_a^b x^{\alpha_n}f(x)dx=0.$$

Aux moyen d'un changement convenable de la variable et de la fonction, cette hypothèse revient alors à supposer qu'en posant, dans le cas de $a\neq 0$, $e^t=(x/a)^\beta$, $T=\log(b/a)^\beta$,

$$a_k(\xi, t)=e^{\xi t}\int_0^T e^{-\xi\tau}\varphi(\tau)d\tau \quad (k \text{ entier positif}),$$

on a $\lim_{t\to\infty}a_k(\xi, t)=0$. En appliquant une identité de Phragmén [Acta Math. **28**, 351–368 (1904)] on déduit que, pour tous les valeurs de t, $0<t<T$, $\int_0^t\varphi(\tau)d\tau=0$ et par suite la fonction sous le signe est nulle presque-partout. Le cas où $a=0$ se résout par un petit détour. *B. Levi.*

San Juan, Ricardo. Some noteworthy asymptotic developments. Revista Mat. Hisp.-Amer. (4) **11**, 55–110 (1951). (Spanish)

Let the origin O be an interior or boundary point of a given region R throughout which $f(z)$ is defined. Following Stieltjes the author states that if $f(z)=\sum_{\nu=0}^{n-1}a_\nu z^\nu+z^nR_n(z)$, where $R_n(z)\to a_n$ as $z\to 0$ for every n, then $f(z)$ admits the asymptotic expansion $\sum_{\nu=0}^{\infty}a_\nu z^\nu$ in R. If $f(z)$ is continuous in the bounded region R, then, developing Poincaré's ideas, the author gives as an equivalent definition the boundedness of the quotients $|f(z)-\sum_{\nu=0}^{n-1}a_\nu z^\nu|/|z^n|$ $(<m_n)$ for all given n, and z in R. The m_n he terms "bounds of Poincaré" and he considers also the bounds $m_n^{(p)}$ of order p $(m_n\equiv m_n^{(0)})$ defined by the inequalities $|f(z)-\sum_{\nu=0}^{n-1}a_\nu z^\nu|/|z^{n-p}|<m_n^{(p)}$. Relations between the different bounds are exhibited when R is a circle of radius r about O.

The author is concerned with obtaining asymptotic expansions arising in certain moment problems, determining their region of validity, and the bounds m_n in most cases, and $m_n^{(1)}$, $m_n^{(2)}$, in addition, in others. The simple case of a power series is first dealt with and this is the only case in which the expansion is about an interior point of R. The Stieltjes integral $\int_0^\infty(1+tz)^{-1}d\psi(t)$, the asymptotic expansion of which is valid over the interior of the whole plane cut by the negative real axis, forms the first of a series in which R is an angle and O a boundary point (not necessarily the origin). The immediate extension to $\int_0^\infty(1+tz^h)^{-1}d\psi(t)$ gives a region of validity of $|\arg z|<\pi/h$. More general moment problems follow in which the asymptotic expansion is valid, first in an angle bisected by the positive real axis, and later in the case of any given angle. In the most general cases the asymptotic expansion of $\int_0^{\infty(\rho)}\alpha(t)\Phi(tz)dt$ in powers of z^{-1} is shown to be valid in the interior of a certain angular region.

The bounds are described in terms of integrals involving the moment function $\Phi(t)$ and its primitives and in terms of the kernel $\alpha(t)$ and its derivatives.

The statements of the theorems are too long to be reproduced here. While of a general character they involve conditions on both the moment function and the kernel. Detailed applications are given to the case of the Laplace transformation, making use of the results of Gorny [Acta Math. 71, 317–358 (1939); these Rev. 1, 137] and of the methods of Boas and Pólya [Duke Math. J. 9, 406–424 (1942); these Rev. 3, 292] and to Mittag-Leffler's generalisation of this. The paper contains a large number of misprints and a few minor errors which make some of the proofs difficult to follow. *R. Wilson* (Swansea).

Calculus

Court, L. M. A theorem on conditional extremes with an application to total differentials. Proc. Amer. Math. Soc. 2, 423–428 (1951).

For extrema of point functions on a set defined by a finite number of constraints of a special type the author obtains a reciprocity theorem in regard to the first and second order conditions, and shows how the results may be applied in the consideration of total differentials. In particular, the results of the present paper extend greatly those of an earlier paper of the author [J. Math. Physics 20, 99–106 (1941); these Rev. 2, 287]. *W. T. Reid* (Albuquerque, N. M.).

Grosswald, Emil. On the integration scheme of Maréchal. Proc. Amer. Math. Soc. 2, 706–709 (1951).

Integrations of a continuous function over a two-sphere and its interior are carried out as limits of line integrals over spirals on the sphere and on concentric spheres, respectively. For the circle, the corresponding theorem was proved by Wilkins [Bull. Amer. Math. Soc. 55, 191–192 (1949); these Rev. 10, 516]. *P. Civin* (Eugene, Ore.).

Hsu, L. C. The asymptotic behavior of an integral involving a parameter. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 273–279 (1949).

Let the real function $f(x, y)$ be continuous together with $f_x(x, y)$ and $f_{xx}(x, y)$ in the half-strip $a \leqq x \leqq b$, $N < y < \infty$ (where a may be $-\infty$ and where b may be $+\infty$) such that (i) $f_x(x, y) = 0$ at some point of the half-strip and $f_{xx}(x, y) < 0$ everywhere in the half-strip, (ii) between a and b there is a number ξ with $f_x(\xi, y) \to 0$ and $y \to \infty$, (iii) $f_{xx}(x, y) \to -\infty$ and $f_{xx}(x, y)/f_{xx}(\xi, y) \to 1$ as $(x, y) \to (\xi, \infty)$. Then the integral $\int_a^b \exp(f(x, \lambda))dx$ is for large positive λ asymptotically equal to

$$\exp(f(\xi, \lambda))\{-2\pi/f_{xx}(\xi, \lambda)\}^{\frac{1}{2}}.$$

This theorem contains as special case the classical asymptotic formula of Laplace, in which $f(x, \lambda) = \lambda f(x)$.

J. G. van der Corput (Stanford University, Calif.).

Clemmow, P. C. Some extensions to the method of integration by steepest descents. Quart. J. Mech. Appl. Math. 3, 241–256 (1950).

If $f(\lambda)$ is indefinitely differentiable for $\lambda \geqq 0$, the integral

$$J_1 = \int_0^\infty f(\lambda) e^{-k\lambda^3} d\lambda$$

possesses under certain conditions for large $|k|$ an asymptotic expansion of the form (1) $f_0(k) + f_1(k) + \cdots$, where

$$f_s(k) = \frac{1}{2} \frac{f^{(s)}(0)}{s!} \left(\frac{s-1}{2}\right)! k^{-\frac{1}{2}s-\frac{1}{2}}.$$

If $|2ka^2|$ is large, integration by parts gives for the integral

$$J_2 = e^{ka^3} \int_a^\infty f(\lambda) e^{-k\lambda^3} d\lambda = \int_0^\infty f(\lambda + a) e^{-2ka\lambda - k\lambda^3} d\lambda$$

an asymptotic expansion according to ascending powers of $(2ka^2)^{-1}$. But expansion of $f(\lambda + a)$ into ascending powers of λ furnishes for J_2, even if a is small, an asymptotic expansion of the form (1), where

$$f_s(k) = \frac{f^{(s)}(a)}{s!} \int_0^\infty \lambda^s e^{-2ka\lambda - k\lambda^3} d\lambda;$$

the leading term $f_0(k)$ may be written as an error integral or a complex Fresnel integral. If $|2ka^2|$ is large, $f_s(k)$ can be expanded asymptotically according to ascending powers of $|2ka^2|^{-1}$.

If in the integral J_1 the function $f(\lambda)$ possesses singularities which lie near the origin, it may be useful to write $f(\lambda) = g(\lambda)h(\lambda)$, where $h(\lambda)$ contains these singularities. Thus we obtain under certain conditions for J_1 the asymptotic expansion (1), where

$$f_s(k) = \frac{g^{(s)}(0)}{s!} \int_0^\infty \lambda^s h(\lambda) e^{-k\lambda^3} d\lambda.$$

In the special case $h(\lambda) = (\lambda^2 + z)^{-p}$, where p is real, $f_s(k)$ can be expressed by means of a confluent hypergeometric function. In several of his expansions the author gives an estimate for the remainder. In the last section he treats two physical applications, first diffraction by a "black" screen and finally radio propagation over a finitely conducting, plane earth. *J. G. van der Corput.*

Šelkovnikov, F. A. The generalized Cauchy formula. Uspehi Matem. Nauk (N.S.) 6, no. 3(43), 157–159 (1951). (Russian)

A proof of the following formula is given:

$$\int_a^t \varphi(t_1)dt_1 \int_a^{t_1} \varphi(t_2)dt_2 \cdots \int_a^{t_{n-1}} \varphi(t_n)dt_n \int_a^{t_n} f(\tau)d\tau = \frac{1}{n!} \int_a^t f(\tau) \left(\int_\tau^t \varphi(\xi)d\xi \right)^n d\tau.$$

Moon, Parry, and Spencer, Domina Eberle. Cylindrical and rotational coordinate systems. J. Franklin Inst. 252, 327–344 (1951).

***Anđelić, Tatomir P. Teorija vektora.** [Theory of Vectors]. Izdavačko Preduzeće Srbije, Belgrade, 1947. viii+408 pp.

Vujaklija, Gojko. Une démonstration des deux théorèmes connus de l'algèbre vectoriel. Bull. Soc. Math. Phys. Serbie 2, nos. 3–4, 47–49 (1950). (Serbo-Croatian. French summary)

Athen, H. Vektoren auf der Kugelfläche. Z. Angew. Math. Mech. 31, 186–188 (1951).

By defining a vector on a sphere as an arc of a great circle with a sense of description, and by postulating certain rules of operation, the author deduces the formulae of spherical trigonometry. *L. M. Milne-Thomson* (Greenwich).

Theory of Sets, Theory of Functions of Real Variables

O*Banach, Stefan. Wstęp do teorii funkcji rzeczywistych. [Introduction to the theory of real functions]. Monografie Matematyczne. Tom XVII. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951. iv+224 pp.

The book contains the first part of a manuscript of Banach written before World War II, and planned as a two volume text on the theory of functions of real variables. The first part was in print in Cracow at the outbreak of the war but both the type and the manuscript were destroyed by the German occupation authorities. A copy of the first 10 proof sheets survived, however, and is now edited by Orlicz and Alexiewicz in the present volume.

The contents are as follows. Ch. I: Abstract set theory (includes a brief introduction to mathematical logic). Ch. II: Limits of sequences. Ch. III: Point-sets (a study of fundamental topological properties of sets contained in the n-dimensional euclidean space E^n). Ch. IV: Functions on E^n (includes a discussion of curves in E^n). Ch. V: Riemann's integral (includes multiple integrals and the theory of Jordan's measure).

The presentation is distinguished by the conciseness and elegance, characteristic of Banach. *S. Ulam.*

✓*Jeffery, R. L. The Theory of Functions of a Real Variable. Mathematical Expositions, no. 6. University of Toronto Press, Toronto, Ont., 1951. xiii+232 pp. $6.00.

The author states that his main purpose in this book is to present the contents of chapter VI (The inversion of derivatives) and of chapter VII (Derived numbers and derivatives) (pp. 140–203). The former includes, in addition to standard theorems concerning absolutely continuous functions and the integrals of finite summable derivatives, an interesting discussion of the problem of determining a continuous function from its finite (but not summable) derivative by a denumerable infinity of operations: the chapter is a useful introduction to the Denjoy integral. In chapter VII properties of the non-differentiable functions of Weierstrass and of Besicovitch are established, followed by an exhaustive analysis of the distribution of the derivates and approximate derivates of an arbitrary function of one variable.

The first five chapters give an introduction to the theory of measure and the Lebesgue integral, the methods being fundamentally those of Lebesgue. The idea of "metrically separated" sets is used in defining measurability, and a proof of Birkhoff's ergodic theorem is given in chapter IV. The presentation of the classical theory, although complete, is intentionally brief, and perhaps for that reason seemed to the reviewer to be less clear and precise than that of chapters VI and VII: the young graduate student (for whom these chapters are principally intended) may find some parts of the theory difficult to understand without assistance. For example, the nature of the limit process in the definitions of the Lebesgue and Riemann integrals (pp. 67–75) is not fully explained, and Definition 3.4 might even lead the reader to suppose that $x^{-1}\sin(x^{-1})$ is integrable in the Lebesgue sense over an interval $(0, a)$, in spite of later explanations (p. 146). The final chapter (VIII) deals briefly with the Riemann-Stieltjes integral and linear functionals. Theorem 8.1 (stated without proof) suggests an analogy with the Riemann integral which is false unless $g(x)$ is monotone; even if the condition in the theorem is strengthened to the form $\sum(M_i-m_i)|g(x_i)-g(x_{i-1})|$ some additional hypothesis is still required, for example, that $f(x)$ or $g(x)$ is of bounded variation. *U. S. Haslam-Jones* (Oxford).

Seshadri, Patta. To construct an alef-one set. Math. Student **18** (1950), 13–16 (1951).

The set in question has power $\aleph_1$, and its elements are ordinary sequences of natural numbers. Cf. Hausdorff [Ber. Verh. Sächs. Akad. Wiss. Leipzig **59**, 84–159 (1907), pp. 155–156]. *F. Bagemihl* (Rochester, N. Y.).

Cuesta, N. Structures and the Erlangen program. Revista Mat. Hisp.-Amer. (4) **11**, 17–51 (1951). (Spanish)

A discussion, hard to summarize and with many illustrations, of the classification of the subsets of a general abstract system S, on which n groups of "automorphisms" (with respect to different properties) operate independently. The case $n=1$ gives the abstract setting of Felix Klein's Erlangerprogramm. The author criticizes the point of view of M. Dolcher [Rend. Sem. Mat. Univ. Padova **18**, 265–291 (1949); these Rev. **11**, 238]. *G. Birkhoff.*

Arnold, B. H. Birkhoff's problem 20. Ann. of Math. (2) **54**, 319–324 (1951).

Consider the relation ρ between subsets X of a space E, and convergent directed sets $\{x_\alpha\}\to a$, defined to mean: if X contains a cofinal subset of $\{x_\alpha\}$, then X contains a. This relation has the remarkable property that the polarity [G. Birkhoff, Lattice Theory, 2d ed., p. 55, Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. **10**, 673] associated with ρ gives precisely the usual correspondence between closed subsets and convergent directed sets [see J. Tukey, Convergence and uniformity in topology, Chap. III, Princeton Univ. Press, 1940; these Rev. **2**, 67]. The proof of this fact by the author solves Problem 20 of the reviewer's book, named above. Various other related results are also obtained. *G. Birkhoff* (Cambridge, Mass.).

Sikorski, R. On the existence of the generalized limit. Studia Math. **12**, 117–124 (1951).

Let A be a directed system; an A-sequence x is a function defined on A with values in a topological space X. Let $L(x)$ be the set of cluster points of x, that is, $L(x)=\{y \mid y\in X$ and for each neighborhood U of y and each $\alpha\in A$ there exists $\beta>\alpha$ such that $x_\beta\in U\}$; let $C(x)$ be the closure in X of the set of values $\{x_\alpha \mid \alpha\in A\}$. Then x is called a compact A-sequence if $C(x)$ is a compact Hausdorff space.

This paper contains a definition, based on the Čech compactification B of A regarded as a space with the discrete topology, of a generalized limit for every compact A-sequence. Let i be the A sequence defined by the identity mapping of A into B, and let β_0 be a point of $L(i)$ not in A. Then every compact A-sequence x can be regarded as a continuous image of A under the mapping $\alpha\to x_\alpha$ of A into $C(x)$; this mapping has a unique extension $\beta\to x(\beta)$ carrying all of B into $C(x)$. Let $\operatorname{Lim} x_\alpha = x(\beta_0)$. Theorem 1. For compact A-sequences, $\operatorname{Lim} x_\alpha$ has the following properties: (1) $\operatorname{Lim} x_\alpha \in L(x)$; hence if x is convergent $\operatorname{Lim} x_\alpha = \lim x_\alpha$. (2) If $x_\alpha = y_\alpha$ for all $\alpha>\alpha_0$, then $\operatorname{Lim} x_\alpha = \operatorname{Lim} y_\alpha$. (3) If S is a set of indices and if, for each $s\in S$, x^s is a compact A-sequence in a topological space X^s, and if F is a continuous mapping of the topological product $P_{s\in S}C(x^s)$ into a Hausdorff space, then $F(\{x^s\})$ is a compact A-sequence and $\operatorname{Lim} F(\{x^s\}) = F(\{\operatorname{Lim} x^s\})$; consequently Lim commutes with continuous algebraic operations such as addition in a topological group. This is used to prove existence of an

invariant average for functions from an Abelian semigroup into compact convex subsets of linear topological spaces.

M. M. Day (Urbana, Ill.).

Nöbeling, Georg. Über eine Verallgemeinerung des Folgenbegriffes. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 133–141 (1951).

A discussion of the merits of the concepts of "filter" and "base de filtre" (alias Raster), introduced by H. Cartan.

G. Birkhoff (Cambridge, Mass.).

Sierpiński, W. L'opération du crible et les fonctions analytiques d'une suite infinie d'ensembles. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **41** (1948), 47–62 (1950). (French. Polish summary)

Une fonction univoque $f(E_1, E_2, \cdots)$ d'une suite infinie quelconque d'ensembles linéaires et dont la valeur est toujours un ensemble linéaire, est dite [Kantorovitch et Livenson, Fund. Math. **18**, 214–279 (1932), pp. 224–225] "fonction analytique" si les hypothèses $a \varepsilon f(E_1, E_2, \cdots)$ et b non-$\varepsilon f(H_1, H_2, \cdots)$, a et b étant deux nombres réels, entraînent toujours l'existence d'un entier $k>0$ tel que ou bien $a \varepsilon E_k$ et b non-εH_k, ou bien a non-εE_k et $b \varepsilon H_k$. Une fonction analytique f étant donnée, on appelle F_f la famille de tous les ensembles $f(E_1, E_2, \cdots)$ où les E_n $(n=1, 2, \cdots)$ sont des ensembles linéaires fermés. Q étant un ensemble plan et π étant une propriété qu'un ensemble linéaire peut avoir ou ne pas avoir, on appelle $\Gamma_\pi(Q)$ l'ensemble de tous les nombres réels a tels que l'intersection de la droite $x=a$ avec Q ait la propriété π. Une propriété π d'ensemble linéaire étant donnée, on appelle Φ_π la famille de tous les ensembles $\Gamma_\pi(Q)$ où Q est un ensemble plan fermé quelconque.

L'auteur démontre les théorèmes suivants. Théorème 1: Pour toute fonction analytique f, il existe une propriété π d'ensemble linéaire telle que $\Phi_\pi = F_f$. Théorème 2: Pour toute propriété π d'ensemble linéaire, il existe une fonction analytique f telle que $\Phi_\pi = F_f$. Théorème 3: La famille de toutes les fonctions analytiques est la plus petite famille **F** de fonctions univoques d'une suite infinie d'ensembles linéaires ayant pour valeur un ensemble linéaire et jouissant des trois propriétés suivantes: (1) Toute fonction $f_k(E_1, E_2, \cdots) = E_k$ $(k=1, 2, \cdots)$ appartient à **F**. (2) Si $f \varepsilon \mathbf{F}$, alors $X - f \varepsilon \mathbf{F}$ (X étant l'ensemble de tous les nombres réels). (3) La réunion d'un ensemble quelconque de fonctions appartenant à **F** appartient à **F**. L'auteur compare les fonctions analytiques avec leur cas particulier: les "fonctions analytiques positives" [Kantorovitch et Livenson, loc. cit.]; ces dernières coïncident [ibid.] avec les fonctions $\delta\sigma$ de F. Hausdorff. Le théorème 2 cesse d'être vrai si, dans son énoncé, on remplace les mots "fonction analytique" par les mots "fonction analytique positive".

A. Appert.

Gehring, Frederick W. Images of convergent sequences in sets. J. London Math. Soc. **26**, 249–256 (1951).

A set S' in E^n is called an image of S if there is a similarity transformation which carries S into S'. Hadwiger [Comment Math. Helv. **19**, 236–239 (1947); these Rev. **8**, 255] has shown that if E has positive interior measure, then it contains an image of any finite set S. The author studies this for infinite S, and in particular for S a convergent sequence. If E is nowhere dense, then there is a convergent sequence no image of which lies in E. If E has positive interior measure, all sufficiently rapidly converging sequences have images in E, but there is no non-trivial sequence which has images in every such set E. Kestelman [J. London Math. Soc. **22**, 130–136 (1947); these Rev. **9**, 274] has called E universal if every convergent sequence has a translate which, except for a finite number of terms, lies in E. If E is universal, it contains an image of every convergent sequence, but not conversely.

R. C. Buck (Madison, Wis.).

Gustin, William. Partitioning an interval into finitely many congruent parts. Ann. of Math. (2) **54**, 250–261 (1951).

An interval is a non-null, bounded, convex subset of a dense additive subgroup of real numbers, and a partition of it is a proper decomposition into a finite number of subsets, called parts, which are disjoint and congruent by pairs. Congruence includes reflection as well as translation. For an interval which, if symmetric contains its midpoint, the author proves that (i) every part of a partition is a finite union of subintervals, (ii) if the interval can be partitioned into n parts it can also be partitioned into n subintervals. The author gives a series of definitions culminating in the definition of the regularity of a partition. He then shows that every partition of an interval of the type given above is necessarily regular and that a regular partition has property (i). From (i) he deduces (ii) by using the fact that the length of the interval is equal to the sum of the lengths of its parts and by considering the various cases that can arise. Rather more than this is proved and the theorems are stated for "unsemisymmetric partitions". It is not known whether every interval that can be semisymmetrically partitioned into n parts can be partitioned into n subintervals.

H. G. Eggleston (Swansea).

Tsuji, Masatsugu. On Baire's theorem concerning a function $f(x, y)$, which is continuous with respect to each variable x and y. J. Math. Soc. Japan **2**, 210–212 (1951).

The author gives a simple proof of the following theorem due to R. Baire [Ann. Mat. Pura Appl. (3) **3**, 1–122 (1899)]. Let $f(x, y)$ be continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, in each variable separately. Then there is a set X on the x-axis that is dense on $[0, 1]$ and such that $f(x, y)$ is continuous in x and y jointly at any point (x_0, y_0) with $x_0 \varepsilon X$ and $0 \leqq y_0 \leqq 1$: and similarly for the y-axis.

H. P. Mulholland (Birmingham).

Pettis, B. J. Uniform Cauchy points and points of equicontinuity. Amer. J. Math. **73**, 602–614 (1951).

The author considers directed systems of continuous functions from a topological space X into a uniform space Y. Let X be any topological space and ϕ any cardinal. A subset E of X is a I_ϕ-set if it is the union of ϕ nowhere dense sets, and otherwise is a II_ϕ-set; it is ϕ-residual if its complement is a I_ϕ-set. Let Y be any completely regular topological space, not necessarily Hausdorff, and of all uniformizations of the topology in Y let $\{V_\alpha | \alpha \varepsilon A\}$ be one having least cardinal $|A|$. For each element λ of a fixed directed set Λ let f_λ be a function on X to Y. A point x_0 in X is a Cauchy point of $\{f_\lambda\}$ if for each $\alpha \varepsilon A$ there is a $\lambda_\alpha \varepsilon \Lambda$ such that $f_\lambda(x_0) \varepsilon V_\alpha(f_{\lambda_\alpha}(x_0))$ for all $\lambda \geqq \lambda_\alpha$; and x_0 is a uniform Cauchy point of $\{f_\lambda\}$ if for each $\alpha \varepsilon A$ there is a $\lambda_\alpha \varepsilon \Lambda$ and an open set G about x_0 such that $f_\lambda(x) \varepsilon V_\alpha(f_{\lambda_\alpha}(x))$ for all $\lambda \geqq \lambda_\alpha$ and all $x \varepsilon G$. The set of Cauchy points is denoted by C and the set of uniform Cauchy points by C_u. A point of almost equicontinuity of $\{f_\lambda\}$ is a point x_0 such that for each $\alpha \varepsilon A$ there is a $\lambda_\alpha \varepsilon \Lambda$ and an open set G about x_0 such that $f_\lambda(x) \varepsilon V_\alpha(f_\lambda(x_0))$ for any $\lambda \geqq \lambda_\alpha$ and any $x \varepsilon G$. The set of all such points is denoted by C_a. If $|A|$ denotes the least of the cardinals of cofinal subsets of Λ and $\phi \geqq \max(|A|, |\Lambda|)$, then the au-

thor's main results are as follows. If each f_λ is continuous and C is ϕ-residual, then C_u and C_v are ϕ-residual and $C_u \subset C_v$. If each f_λ is continuous and C is a Π_ϕ-set, then $C \cap C_v$ $(=C_u)$ is a Π_ϕ-set. The author then considers various applications of these results. *A. F. Ruston.*

Sikorski, R. On measures in Cartesian products of Boolean algebras. Colloquium Math. 2, 124–129 (1951).

The author presents a definition of Cartesian product for Boolean algebras, equivalent to that of Kappos [Math. Ann. 120, 43–74 (1947); these Rev. 9, 178]. Concerning this concept Kappos has proved that if each factor carries a (finite, non-negative, finitely additive) measure, then the product carries, in a natural way, a unique product measure. Kappos, moreover, asserted that if the factors are countably additive measures on σ-algebras, then the product measure is also countably additive. The author constructs an example proving that the latter assertion is false.

P. R. Halmos (Montevideo).

R. Salinas, Baltasar. On the theory of measure. Revista Acad. Ci. Madrid 42, 465–491 (1948). (Spanish)

The author gives a brief axiomatic presentation of the theory of Jordan and Lebesgue measures in an abstract Boolean σ-algebra. *E. F. Beckenbach.*

Moore, Edward F. Convexly generated k-dimensional measures. Proc. Amer. Math. Soc. 2, 597–606 (1951).

If we denote by $C_n^k(A)$ and $H_n^k(A)$ the k-dimensional Carathéodory and Hausdorff measures respectively of a subset A of the n-space E_n, then $C_n^k(A) \leqq H_n^k(A)$ for any $A \subset E_n$, and various cases are known where equality holds. In the present paper the author gives the first example of a set $A \subset E_3$ with $0 < C_3^2(A) < H_3^2(A) < +\infty$. For any set $A \subset E_3$ the further relation is proved $H_3^2(A) < g C_3^2(A)$, where an evaluation of the constant g is given. Finally the author defines certain classes of measures $\phi(A)$ (convexly generated k-dimensional measures) by means of limiting processes from set functions defined for convex sets only. Carathéodory, Hausdorff and certain Gillespie measures are particular cases of the functions $\phi(A)$. *L. Cesari.*

Dieudonné, Jean. Sur un théorème de Jessen. Fund. Math. 37, 242–248 (1950).

Let I be the unit interval, N the set of positive integers, $P = I^N$ the cube consisting of the cartesian product of the sets $I_n = I$ $(n = 1, 2, \cdots)$. For $J \subset N$, $x \in P$; x_J is the projection of x on I^J. Thus if J, J' are complementary we may write $x = (x_J, x_{J'})$. If we consider Lebesgue measure on each I_n then there is the corresponding product measure μ_J in I^J and we write μ for μ_N. For a function f on P integrable with respect to μ we write $f_J(x)$ for $\int_{J'} f(x_J, x_{J'}) d\mu_{J'}$. A result of Jessen is the following: If J_n is an increasing sequence of finite subsets of N whose union is N then the functions f_{J_n} converge to f almost everywhere on P as $n \to \infty$. Let F be the set of finite subsets of N and let F be directed by inclusion then it is known that the $\lim_J f_J = f$ in the mean, where $\lim_J$ is the lim in the sense of filters or directed sets. In this note an example is constructed to show that it is not necessarily true that $\lim_J f_J(x) = f(x)$ almost everywhere if the limit is taken in the generalized sense of filters. The construction also yields a counterexample to an analogous situation in the theory of differentiation. *N. Dunford.*

Lindgren, Bernard W. An integral on a space of continuous functions. Proc. Amer. Math. Soc. 2, 634–643 (1951).

The author defines an integral over C', the space of real-valued functions $x(t)$ defined and continuous on $0 \leqq t \leqq 1$, and he investigates the close relationship of the integral to the Wiener integral over the space C consisting of those functions of C' which vanish at $t = 0$. Among the results which the author obtains is the following theorem (Theorem 3.1): $F[x]$ is measurable in C' if and only if $F[y+\xi]$ is measurable as a function of y and ξ in $C \otimes E_1$, where E_1 is the real line. He also shows (Theorem 3.2) that if $F[x]$ is measurable on C', then

$$\int_{C'} F[x]dx = \int_{C \otimes E} F[y+\xi]d(y, \xi) = \int_{-\infty}^{\infty} d\xi \int_C F[y+\xi]d_y y$$

whenever either of first two integrands is summable over its space, or whenever the third integrand is summable over C for almost all ξ in E_1 and the repeated integral exists and is finite. He shows that these three conditions are equivalent (and unnecessary if $F[x]$ is of constant sign on C'). Other results are (Theorem 4.3): If $f(u)$ is Borel measurable in E_1, and $\zeta(t)$ is of bounded variation in $[0, 1]$, then $f(\int_0^1 \zeta(t)dx(t))$ is w'-measurable; (Theorem 4.4): The set S of functions of C' which do not satisfy for a given $\gamma < \frac{1}{2}$ the Hölder condition $|x(t) - x(u)| < h|t-u|^\gamma$, for all t and u in $[0, 1]$, for some real h, is of measure zero; and (Theorem 4.5): The set of functions of C' which possess derivatives at at least one point is of measure zero. The author also develops integration formulas and points out that various other results analogous to results obtained for the space C by R. H. Cameron and the reviewer can be obtained for the space C'.

W. T. Martin (Princeton, N. J.).

Weston, J. D. Inequalities for Riemann-Stieltjes integrals. Math. Z. 54, 272–274 (1951).

The generalization of the Arzela theorem, that if $f_n(x)$ is a uniformly bounded sequence of functions converging to a function $f(x)$, whose Riemann-Stieltjes integrals with respect to a monotone non-decreasing function $\alpha(x)$ exist on S, then $\lim_n \int_S f_n d\alpha = \int_S f d\alpha$ where S is an interval or a finite set of non-overlapping intervals, is deduced from the inequalities

$$\overline{\int} (\limsup f_n) d\alpha \geqq \limsup \underline{\int} f_n d\alpha,$$

$$\underline{\int} (\liminf f_n) d\alpha \leqq \liminf \overline{\int} f_n d\alpha.$$

The demonstration avoids a direct use of the Arzela theorem on sequences of finite sets of non-overlapping intervals.

T. H. Hildebrandt (Ann Arbor, Mich.).

Gomes, Ruy Luís. The Dirac function—its mathematical interpretation. I, II, III. Gaz. Mat., Lisboa 11, no. 46, 1–3 (1950); 12, no. 47, 11–13 (1951); 12, no. 48, 8–11 (1951). (Portuguese)

This is an essentially expository article showing that the so-called Dirac function cannot be realized as a bona fide function and giving it an interpretation as a linear functional.

I. E. Segal (Princeton, N. J.).

Loud, W. S. Functions with prescribed Lipschitz condition. Proc. Amer. Math. Soc. 2, 358–360 (1951).

A function $g(t)$ is constructed which satisfies, for every real t, a Lipschitz condition of order precisely α $(0 < \alpha < 1)$

(that is, there exist positive constants K_1, K_2 such that (a) for any t and any Δt, $|\Delta g| = |g(t+\Delta t)-g(t)| < K_1|\Delta t|^\alpha$, and (b) for any t and infinitely many, arbitrarily small Δt, $|\Delta g| > K_2|\Delta t|^\alpha$). The construction follows van der Waerden's example of a continuous non-differentiable function [Math. Z. 32, 474–475 (1930)]: if $g(t, h)$ has period $2h$ in t, and $g(t, h) = h^{-1}|t|$ for $-h \leqq t \leqq h$, then $g(t) = \sum_{n=1}^{\infty} k^{-n\alpha} g(t, k^{-n})$, where $k = 2^{2A}$, A an integer with $2^{2A(1-\alpha)} > 2$. *F. A. Behrend* (Melbourne).

Zahorski, Zygmunt. Sur l'ensemble des racines de l'équation $W(x) = f(x)$. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **41** (1948), 43–45 (1950). (French. Polish summary)

The author proves the following theorem: Let $g(x)$ be a continuous function so that for every x one of the derived numbers of $g(x)$ is $+\infty$ and one of the other derived numbers is $-\infty$. Let $f(x)$ be such that all derived numbers of $f(x)$ are everywhere finite. Assume further that

$$[f(u)-g(u)][f(v)-g(v)] < 0.$$

Then the number of roots of $f(x) = g(x)$, $u < x < v$, has the power of the continuum. The author deduces this result from the following sharper one: Let $G(x)$ be a continuous function so that for every x one of the derived numbers of $G(x)$ is >0 and another one is <0. Assume further $G(u) \cdot G(v) < 0$. Then the number of roots of $G(x) = 0$, $u < x < v$, has the power of the continuum. The author takes $g(x)$ to be the nondifferentiable function of Weierstrass and obtains several applications. *P. Erdös* (Aberdeen).

Racine, C. The Riemann integration of sequences. Bull. Allahabad Univ. Math. Assoc. **14** (1942–1950), 19–22 (1950).

Let $\{S_n(x)\}$ be a sequence of uniformly bounded Riemann integrable functions for which S_n tends to a Riemann integrable function $f(x)$. Then $\int_a^b S_n(x)dx \to \int_a^b f(x)dx$. The author gives a brief and elementary proof of this by using the measure of closed and open sets only.

R. L. Jeffery (Kingston, Ont.).

Tolstov, G. P. Curves allowing a differentiable parametric representation. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 135–152 (1951). (Russian)

Given a plane curve C in parametric form, $x = \xi(\theta)$, $y = \eta(\theta)$, the author finds conditions on this or related parametrizations necessary and sufficient that the curve have a parametric representation with suitable differentiability properties. Say that the curve has a right-hand unit tangent vector at a point $P_0 = P(\theta_0)$ if, as $\theta \to \theta_0$ on the set where $\theta > \theta_0$ and $P(\theta) \neq P(\theta_0)$, τ the limit of the unit vector in the direction from P_0 to $P(\theta)$ exists. (If necessary, the parametrization is modified so that ξ and η have no common interval of constancy.) Theorem 1. In order that C possess a parametrization $x = x(t)$, $y = y(t)$, $a \leqq t \leqq b$, with continuous derivatives such that for every t

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \neq 0$$

it is necessary and sufficient that C have a continuously varying right-hand unit tangent vector. Theorem 2 (3) [4]. In order that C have a parametrization with continuous (bounded) [everywhere finite] derivatives it is necessary and sufficient that (a) C be rectifiable ((a) and (b) if C is parametrized by arc length s, the set of values of s where C possesses no one-sided tangent vector is of Lebesgue measure zero) [C possesses a parametrization whose functions are both of class VBG_* (see Saks, Theory of the Integral, Warsaw, 1937, p. 228)]. The proof of Theorem 3 is different from that of Zahorski [Mat. Sbornik N.S. **22(64)**, 3–26 (1948); these Rev. **9**, 503] to whom the result is due. Other parts of the paper discuss other changes of parameter and settle a question of Zahorski by showing that no curve whose point set has positive plane measure can have an everywhere differentiable parametrization. *M. M. Day.*

Il'in, V. P. Estimates of functions having derivatives summable with a given power on hyperplanes of different dimensions. Doklady Akad. Nauk SSSR (N.S.) **78**, 633–636 (1951). (Russian)

The author gives estimates, too complicated to reproduce here, of bounds for a real-valued function φ defined on an n-dimensional region D in terms of (1) bounds on integrals over D of powers of the absolute value of assorted derivatives of φ, (2) uniform bounds on integrals over all parallel k-dimensional sections of D of powers of other derivatives of φ, (3) constants restricting the shape of D and of the sections, and (4) assorted relations between the various dimensions, orders and powers occurring in the hypotheses. *M. M. Day* (Urbana, Ill.).

Landis, E. M. On the length of the level lines of a function of two variables. Doklady Akad. Nauk SSSR (N.S.) **79**, 393–395 (1951). (Russian)

Let $f(x, y)$ be defined at each point (x, y) of the closed unit square J, and $\nu_f(t)$ be the length (Hausdorff linear measure) of the set E_t of points where $f(x, y) = t$. The author proves, for $p \geqq 2$, that if the pth order derivatives of $f(x, y)$ are continuous everywhere on J then

$$\int_{-\infty}^{\infty} [\nu_f(t)]^p dt < \infty.$$

He notes that the case $p = 1$ is known: in fact, it follows from less restricted results due to the reviewer [Proc. London Math. Soc. (2) **46**, 290–311 (1940), Theorem 1; these Rev. **1**, 303] and to A. S. Kronrod [Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 24–134 (1950), Theorem 33; these Rev. **11**, 648]. *H. P. Mulholland* (Birmingham).

Landis, E. M. On the set of singular points of a differentiable function of several variables. Doklady Akad. Nauk SSSR (N.S.) **79**, 569–572 (1951). (Russian)

Let F be a continuous function defined on an n-dimensional region G, possessing k successive total differentials in G. A critical point of F is a point where grad F vanishes. The contingent of the set Ω of critical points of F at a point a of Ω is the union of the set of rays ρ from a such that there exist $\{a_p\} \subset \Omega$ for which $d(a_p, \rho)/d(a_p, a) \to 0$ as $p \to \infty$ ($d(x, E)$ is the distance from x to E). The author proves that the contingent of Ω at a typical point a of Ω is not contained in any k-dimensional hyperplane; that is, if E is the set of those points of Ω where the contingent is contained in a k-dimensional hyperplane, then the Lebesgue measure of $F(E)$ is zero. The proof uses induction on the dimension n. The special case $k = n$ (where $F(\Omega)$ is of measure zero) was proved by Kronrod and Landis [same Doklady **58**, 1269–1272 (1947); these Rev. **9**, 339]. *M. M. Day.*

Cesari, Lamberto. Sulla rappresentazione delle superficie. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **7** (1949–50), 3–9 (1951).

A résumé of the author's work on the existence of a quasi-conformal representation for a continuous surface [these Rev. **8**, 258, 506; **10**, 362; **11**, 588]. *R. G. Helsel.*

Scott, W. R. The essential multiplicity function. Duke Math. J. **18**, 707–717 (1951).

The author considers the question: when is a function the essential multiplicity function (e.m.f.) of a transformation? The e.m.f. has been defined for a continuous plane transformation by T. Radó and P. Reichelderfer [Trans. Amer. Math. Soc. **49**, 258–307 (1941), especially p. 262; these Rev. **2**, 257], and it is defined similarly by the author for a continuous function $f(x)$ of one variable x with $0 \leqq x \leqq 1$. He shows that a function $g(y)$ defined for $-\infty < y < \infty$ is the e.m.f. of some continuous function $f(x)$ on $[0, 1]$ if and only if (i) the values of $g(y)$ are among $0, 1, 2, \cdots, \infty$, (ii) $g(y)$ is lower semi-continuous, (iii) there are finite constants y_0, y_1, y_2, y_3 with $y_2 \leqq y_0 \leqq y_1 \leqq y_3$, such that $g(y)$ is odd or ∞ for $y_0 < y < y_1$ but even or ∞ otherwise and $g(y) > 0$ for $y_2 < y < y_3$ but $g(y) = 0$ otherwise. He develops a more general definition and theorem in which $f(x)$ need not be continuous, and also obtains a partial answer to his question for continuous transformations in the plane.

H. P. Mulholland (Birmingham).

Mickle, Earl J., and Radó, Tibor. On a theorem of Besicovitch in surface area theory. Rivista Mat. Univ. Parma **2**, 19–45 (1951).

Let $S: z = f(x, y)$, $(x, y) \varepsilon Q = (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$, be any non-parametric continuous surface and let $L(S)$ be the Lebesgue area of S, and $H_2(S)$ the two-dimensional Hausdorff measure of the set of the points of S. H. Federer [Trans. Amer. Math. Soc. **62**, 114–192 (1947); these Rev. **9**, 231], in extending a previous result of A. S. Besicovitch [Quart. J. Math., Oxford Ser. (1) **16**, 86–102 (1945); these Rev. **7**, 282] has proved the equality (1) $H_2(S) = L(S)$. This result was obtained by comparing various definitions of area and particularly one which is a modification of the Favard measure. In the present paper the authors give a remarkably simple proof of the equality (1) by utilizing only Hausdorff and Lebesgue measures. Under the same hypotheses for S the further equality $L(S) = A_B(S)$ is proved, where $A_B(S)$ is the so called Besicovitch area of S. The equality (1) is also extended to a large class of parametric surfaces.

L. Cesari (Lafayette, Ind.).

Theory of Functions of Complex Variables

Ryll-Nardzewski, Czesław. Une extension d'un théorème de Sturm aux fonctions analytiques. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 5–7 (1950). (French. Polish summary)

Sturm's theorem on the minimum distance between the zeros of a solution of the differential equation $y'' + A(x)y = 0$ is equivalent to the theorem that a twice differentiable function $f(x)$, not identically zero, has at most one zero in any open interval S of length π/M if $-f''(x)/f(x) \leqq M^2$ in S. This theorem is generalized in the present paper as follows: Let $f(z)$ ($\not\equiv 0$) be an analytic function of the complex variable z in a convex domain D of diameter π/M and let $|f''(z)/f(z)| \leqq M^2$ in D; then $f(z)$ has at most one zero in D. To prove this theorem, the author assumes that at least two zeros of $f(z)$ lie in D, and chooses two z_1 and z_2 such that the line segment joining z_1 and z_2 has no additional zero of $f(z)$. The real part of the function

$$g(t) = f[z_1 + (z_2 - z_1)t], \quad 0 \leqq t \leqq 1,$$

of the real variable t is then found to satisfy a differential equation of the above type, but, in contradiction to Sturm's theorem, to have a zero at both $t = 0$ and $t = 1$.

M. Marden (Milwaukee, Wis.).

Kegel, Günter. Conditions for monogeneity in a point. Revista Cientifica **1**, no. 3, 11–18 (1950). (Portuguese)

Let $w = f(z)$ have a differential at z_0 and let

$$g(z) = [f(z) - f(z_0)]/(z - z_0).$$

The author shows that the existence of $\lim_{z \to z_0} |g(z)|$ or of $\lim \arg_{z \to z_0} g(z)$ insures the monogenic behavior of $f(z)$ at z_0.

R. C. Buck (Madison, Wis.).

Fine, N. J. Note on the Hurwitz zeta-function. Proc. Amer. Math. Soc. **2**, 361–364 (1951).

The Hurwitz zeta-function is defined for $\Re(s) > 1$ and $0 < a < 1$ by $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$. Its continuation to the entire s-plane with the exception of a simple pole at $s = 1$, and its functional equation are here obtained by using the transformation equation for the theta-function $\vartheta(\pi a, ix)$.

W. H. Simons (Vancouver, B. C.).

Szekeres, G. The asymptotic behaviour of the coefficients of certain power series. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Federico Riesz LXX annos natis dedicatus, Pars B, 187–198 (1950).

The polynomial $A_k = A_k(t)$ defined by

$$\exp [K(z - f(tz))] = \sum_{k=0}^{\infty} A_k z^k$$

depends on k, t, K and the choice of the function

$$f(z) = c_1 z + c_2 z^2 + \cdots.$$

The purpose of this paper is to determine the asymptotic behavior of these polynomials A_k for large positive values of k and K such that kK^{-1} tends to a limit. The fundamental idea of this new method is that $\varphi_k = A_{k-1}/A_k$ satisfies a recurrence relation of the form

$$\varphi_k = \frac{k}{K} + \sum_{h=1}^{\infty} c_h t^h \varphi_k \cdots \varphi_{k-h+1}.$$

Thus the author finds that the continuous root $d = d_k(t)$ of the equation $d = k/K + \sum_{h=1}^{\infty} c_h t^h d^h$, defined by $d_k(0) = k/K$ is a first approximation to φ_k. Continuing in this way he finds an asymptotic expansion for $\varphi_k(t)$ and therefore also for

$$\log (k! K^{-k} A_k) = -K \int_0^t \left(\varphi_k(u) - \frac{k}{K} \right) u^{-1} du.$$

As special case he obtains the asymptotic expansion, found by Plancherel and Rotach [Comment. Math. Helv. **1**, 227–254 (1929)] for the Hermite polynomial $H_k((2k+1)^{\frac{1}{2}} \cosh \varphi)$, where φ is positive. *J. G. van der Corput.*

Eggleston, H. G. A generalization of the Hurwitz composition theorem to irregular power series. Proc. Cambridge Philos. Soc. **47**, 477–482 (1951).

The author's theorem can be stated in the following symmetric form: Let

$$f(z)=\sum_1^\infty \Gamma(\lambda_i+1)d_i/z^{(\lambda_i+1)},\quad g(z)=\sum_1^\infty \Gamma(\mu_j+1)\beta_j/z^{(\mu_j+1)},$$

$$h(z)=\sum_{i,j=1}^{\infty} \alpha_i\beta_j\Gamma(\lambda_i+\mu_j+1)/z^{(\lambda_i+\mu_j+1)},$$

where the first two series have finite radii of absolute convergence R_1, R_2, respectively, and $\{\lambda_i\}$, $\{\mu_j\}$ are real positive increasing sequences tending to infinity. Then the series $h(z)$ has a finite radius of absolute convergence, not larger than R_1+R_2, and the function $h(z)$ may be continued analytically to all points of the z-plane which can be joined by a continuous curve Λ to the point at infinity in such a manner that no point of Λ can be expressed in the form z_0+z_1, where z_0 is a point of the "associated set" of $f(z)$ with respect to a, and z_1 is a point of the "associated set" of $g(z)$ with respect to a, $|a|>R_1+R_2$. Roughly speaking, the "associated set" of $f(z)$ is the complement of the set of points to which an element of $f(z)$ can be analytically continued. It corresponds to the set of singular points in the Hurwitz theorem referred to in the title, which is the case $\lambda_i=\mu_i=1$ of the present theorem. The proof is similar to that of the Hurwitz theorem, but complicated by the multivalued character of the functions.

R. M. Redheffer (Los Angeles, Calif.).

Bagemihl, F. Concerning non-continuable, transcendentally transcendental power series. Proc. Nat. Acad. Sci. U. S. A. **37**, 211–213 (1951).

If $\sum a_n z^n$ has radius of convergence 1, then there are c sign-sequences $\{\epsilon_n\}$ (each $\epsilon=\pm 1$) such that $\sum \epsilon_n a_n z^n$ does not satisfy any algebraic differential equation, and at the same time is not continuable. This is proved by the classical argument Hurwitz-Pólya gave for the corresponding result on continuation alone. Next, it is shown that there are c series, without gaps, which have the above singular behavior for all sign-sequences. (The author's proof could be slightly simplified at one point by noting that his series is an entire function plus a gap series, and using the Ostrowski gap theorem.) *R. M. Redheffer* (Los Angeles, Calif.).

Nassif, M. Overconvergence of simple series. J. London Math. Soc. **26**, 295–297 (1951).

If $f(z)$ is regular in $|z|<R$ and $\{p_n(z)\}$ is a simple set of polynomials effective in $|z|\leqq R'<R$, then if the basic expansion of $f(z)$ in terms of $\{p_n(z)\}$ has Hadamard gaps, the corresponding partial sums have the Ostrowski overconvergence property in a neighborhood of every regular point on $|z|=R$. *R. P. Boas, Jr.* (Evanston, Ill.).

Martin, Yves. L'ultraconvergence dans les séries d'interpolation ou de facultés. Bull. Sci. Math. (2) **75**, 80–91 (1951).

Let $f(z)=\sum_0^\infty a_n P_n(z)$, $P_0(z)=1$, $P_n(z)=\prod_{k=1}^n(1-z/\lambda_k)$, $\lambda_k=\rho_k e^{i\theta_k}$, where $\rho_k\uparrow$, $\rho_k\to\infty$, $\theta_k\to 0$, and

$$R(n)=\sum_{k=1}^{n} 1/\rho_k\to\infty.$$

Suppose $a_n=0$ in the intervals

$$m_k\leqq n\leqq n_k,\quad R(n_k)>(1+\theta)R(m_k),\quad \theta>0.$$

Then the partial sums $\sum_0^n a_n P_n(z)$ converge to $f(z)$ uniformly about every point of analyticity of $f(z)$ on the line bounding the half-plane of convergence of the series $\sum a_n P_n(z)$. This theorem and others analogous to familiar results on convergence of subsequences of partial sums of Taylor's series [cf. G. Bourion, L'ultraconvergence dans les séries de Taylor, Actualités Sci. Ind. no. 472, Hermann, Paris, 1937, pp. 8 ff.] illustrate the relation between overconvergence and lacunary structure for series of interpolation of Newton. The well-known theorem of Jentzsch concerning the zeros of the partial sums of Taylor's series has its analogue here also. Similar results are obtained for series of the type $\sum_1^n a_n R_n(z)$, $R_n(z)=1/\prod_{k=1}^n(1+z/\lambda_k)$, $\lambda_k=\rho_k e^{i\theta_k}$, $\rho_k\uparrow$, $\sum_1^n 1/\rho_k=\infty$, $\theta_k\to 0$.

The methods of Erdös and Piranian [Duke Math. J. **14**, 647–658 (1947); these Rev. **9**, 232] used to study uniform convergence on the circle of convergence of subsequences of partial sums of Taylor's series are here employed in connection with series of interpolation of the type $g(z)=\sum_{n=0}^\infty a_n Q_n(z)$, $Q_0(z)=1$, $Q_n(z)\prod_{k=1}^n(z-\lambda_k)$, $\lambda_k\to 0$, $\sum_1^n|\lambda_k|<\infty$. For example, if the coefficients a_n are bounded, $a_n=0$ in $m_k\leqq n\leqq n_k$, then $\{\sum_0^{n_k} a_n Q_n(z)\}$ converges to $g(z)$ uniformly on every closed arc of analyticity of $g(z)$ on the circle of convergence provided the lengths of the gaps increase indefinitely. The possibility that the circle of convergence is a natural boundary of the function $g(z)$ is considered in this situation and also in the case in which $\sum_1^n|\lambda_k|=\infty$. *E. N. Nilson* (Hartford, Conn.).

Ullman, J. L. Hankel determinants whose elements are sections of a Taylor series. I. Duke Math. J. **18**, 751–756 (1951).

Es sei $f(z)=\sum_0^\infty a_k z^{-k-1}$ in $|z|<R$ $(0\leqq R<\infty)$ regulär, $s_n(z)=\sum_0^{n-1} a_k z^{-k-1}$ und $S_{n,p}(z)=\|s_{n+i+j-2}(z)\|_1^{p+1}$ die Hankel'sche Determinante mit dem Glied $s_{n+i+j-2}$ in der $i$ten Zeile und $j$ten Spalte. Es sei ferner Σ_p der Radius jenes Kreises um 0, in dessen Äussern $f(z)$ meromorph ist und die Gesamtordnung der Pole p nicht übersteigt. Verf. beweist, dass die Nullstellen von f in $|z|>\Sigma_p$ Häufungspunkte der Nullstellen von $S_{n,p}(z)$ für $n\to\infty$ sind; in beliebig kleinen Umgebungen um eine r-fache Nullstelle von f befinden sich für genügend grosse n genau r Nullstellen von $S_{n,p}(z)$. Dies verallgemeinert ein bekanntes Resultat von Hurwitz für $p=0$. Ist die Gesamtordnung der Pole in $|z|>\Sigma_p$ genau gleich p, so ist jeder Punkt von $|z|=\Sigma_p$ Häufungspunkt von Nullstellen von $S_{n,p}(z)$ für $n\to\infty$. Dies gibt für $p=0$ ein Theorem von Jentzsch. Zum Beweis werden die Sätze Hadamards über die Hankel'schen Determinanten, gebildet aus den Koeffizienten einer Potenzreihe, verwendet.

A. Pfluger (Zürich).

Dufresnoy, J., et Pisot, Ch. Prolongement analytique de la série de Taylor. Ann. Sci. École Norm. Sup. (3) **68**, 105–124 (1951).

This is an expository article on the connection between entire functions of exponential type and power series in the unit circle. Rather brief proofs of several theorems on entire functions and on gap power series are given. Unfortunately, references to the relevant literature are almost entirely lacking. The authors appear not to know Carleman's theorem [Ark. Mat. Astr. Fys. **17**, no. 9 (1922); cf. also Levinson, Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940, p. 243; these Rev. **2**, 180]. *R. P. Boas, Jr.* (Evanston, Ill.).

Sunyer Balaguer, F. On the distribution of values of an entire function represented by a lacunary Dirichlet series. Revista Acad. Ci. Zaragoza (2) **5**, no. 2, 25–73 (1950). (Spanish)

Can be regarded as a generalisation to Dirichlet series of results due to the same author [Mem. Real. Acad. Sci. Art. Barcelona (3) **29**, 475–516 (1948); these Rev. **12**, 88] on power series. Ritt's theory of order [Amer. J. Math. **50**, 73–86 (1928)] is employed extended to include infinite order. A typical result is as follows. When $F(s)=\sum_0^\infty a_n e^{-\lambda_n s}$ is of finite order ρ, the series being convergent in the whole plane and $\liminf(\lambda_{n+1}-\lambda_n)=h>0$, then if $f(s)$ and $g(s)$ are of lower order, the number of zeros of $f(s)F(s)-g(s)$ cannot be exceptional (in a sense analogous to that of Borel) in the strip $\alpha<\Im(s)<\alpha+\beta$ if $\beta>\rho$ provided the maximal density of the λ_n is less than some $\Delta=\Delta(\rho,\beta)$ or provided the upper density is less than some $\Delta=\Delta(\rho,\beta,h)$.

A. Macintyre (Aberdeen).

Korevaar, Jacob. The zeros of approximating polynomials and the canonical representation of an entire function. Duke Math. J. **18**, 573–592 (1951).

Let R be a closed set in the z-plane, $C(R)$ the class of those entire functions $f(z)\not\equiv 0$ for which there exist polynomials $f_n(z)$ tending to $f(z)$, as $n\to\infty$, uniformly in any bounded domain and with all zeros in R. The author treats this problem: to characterize $C(R)$, using the properties of R. He had done this already in the first part of his thesis [Approximation and interpolation applied to entire functions, University of Leiden, 1949; these Rev. **10**, 694]. Many of the results are, as he states, not new. The conditions on the $f_n(z)$, however, are simplified, and the set $\varepsilon^{-kp}(z-z_0)^k$ ($z\in R$; k a positive fixed integer) is introduced to give better results. For comparison, and also for references, see the very detailed review cited.

Results due to Laguerre cover the case when R is a line or half-line, to Pólya and Obrechkoff the cases when R is a sector of aperture $<\pi$ and $=\pi$, respectively; R is always supposed to be unbounded. In §2 an extension of Hurwitz's Theorem on the zeros of an analytic function, and results on the approximation by polynomials $g_n(z)=\prod_p(1-z/z_{np})$ are given; in §3 conditions on R which restrict the order of the $f(z)\in C(R)$ (see theorems 3.1, 3.2); in §4 conditions which imply that $\exp(a_k z^k)$ (k as above) belongs to $C(R)$ for any a_k provided that the $k-1$ functions $\exp(a_j z^j)$, $j=1, 2, \cdots, k-1$, belong to $C(R)$ for every set of values $a_1, a_2, \cdots, a_{k-1}$. In §5 regular, in §6 non-regular sets are investigated; R is said to be regular if $C(R)$ is exactly the class of entire functions whose zeros belong to R; it is, therefore, regular if and only if $\exp(az^k)\in C(R)$ for any k (k as above) and any a. Special cases of regular sets R are a sector of aperture $>\pi$, a pair of vertical angles $>\frac{1}{2}\pi$; of non-regular sets: a half-line, half-strip, sector of aperture $<\pi$. In §7 sets of half-plane or strip structure are discussed. The explicit form of the $f(z)\in C(R)$ is given in a number of cases (see §§3, 6, 7). In §8, extending results due to Pólya, Weisner and Szász, the author proves: Let $N(S, n)$ be the number of zeros of the polynomial $f_n(z)=\sum_{k=0}^n a_k z^k$ in a sector S with vertex 0, $N(S, n)=o(n)$ $(n\to\infty)$; then $f(z)=\sum_{k=0}^\infty a_k z^k$ is an entire function of order zero.

H. Kober (Birmingham).

Mergelyan, S. N. On some fundamental questions of the theory of best approximation of functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) **79**, 731–734 (1951). (Russian)

Let E be a compact connected set in the complex plane, with a connected complement, and $A(E)$ the class of functions continuous on E and regular at all interior points of E. Let $\rho_n(f, E)$ denote the best approximation to $f(z)$ by polynomials of degree not exceeding n. The author states without proof a number of positive and negative theorems on bounds for $\rho_n(f, E)$ in terms of properties of f and of E. We quote some typical results. Let L_h be the level lines of the Green's function of CE; Γ, the boundary of CE; $d(\zeta, h)$ the distance from $\zeta\in\Gamma$ to L_h, $d(h)=\max d(\zeta, h)$. Then if E does not separate the plane, $\rho_n(f, E)<c\omega[d(n^{-1})]$, c independent of n, where ω is the modulus of continuity of $f(z)$ on E. If E is a Jordan arc with continuously turning tangent then $\rho_n(f, E)<c(\epsilon)n^{-k+\epsilon}\omega_k(n^{-1+\epsilon})$, $\epsilon>0$, where ω_k is the modulus of continuity of $f^{(k)}(z)$. If $\lambda_n\downarrow 0$ there exist a region D with a rectifiable boundary having a tangent at every point and an $f(z)\in A(\bar{D})$ satisfying a Lipschitz condition of order less than 1, while $\rho_n(f, \bar{D})>\lambda_n$ for large n. *R. P. Boas, Jr.*

Grunsky, Helmut. Über beschränkte Funktionen. Jber. Deutsch. Math. Verein. Abt. 1. **55**, 4–9 (1951).

Report on work done during the last decade on bounded analytic functions in multiply-connected regions.

M. Heins (Providence, R. I.).

Nehari, Zeev. Bounded analytic functions. Bull. Amer. Math. Soc. **57**, 354–366 (1951).

The author gives a clear and valuable account of the present state of the theory, which aims at extending results, such as Schwarz's lemma and interpolation problems, which are classical for the unit circle and so for simply connected domains, to multiply connected domains. The emphasis here is not so much on numerical inequalities, but on a characterisation of the extremal functions, and on discovering relations between these and other domain functions such as kernel functions and Green's function.

W. K. Hayman (Exeter).

Pokornyĭ, V. V. On some sufficient conditions for univalence. Doklady Akad. Nauk SSSR (N.S.) **79**, 743–746 (1951). (Russian)

Let $w=f(z)$ be regular and set $p(z)=\{w, z\}/2$, where $\{w, z\}$ is the Schwarzian derivative. Set $P(r)=\max_{|z-z_0|=r}|p(z)|$, and let $y(r)$ be a solution of the differential equation $y''+P(r)y=0$. The following theorems are stated without proof. (A) If $r=0$ and $r=2R$ are consecutive zeros of $y(r)$, then $f(z)$ is univalent in $|z-z_0|\leqq 2R/\sqrt{5}$. (B) If $r=0$ and $r=R_0$ are respectively a zero of $y(r)$, and the nearest zero of $y'(r)$, then $f(z)$ is univalent in $|z-z_0|\leqq R_0/\sqrt{2}$. (C) If for a certain fixed λ $(0\leqq\lambda\leqq 2)$ and for all z in $|z|<1$

$$|\{w, z\}|\leqq\begin{cases}\dfrac{2^{2\lambda-1}\pi^{2(1-\lambda)}}{(1-|z|^2)^\lambda}, & \text{for } 0\leqq\lambda\leqq 1,\\[2ex] \dfrac{2^{3-\lambda}}{(1-|z|^2)^\lambda}, & \text{for } 1\leqq\lambda\leqq 2,\end{cases}$$

then $f(z)$ is univalent in $|z|<1$. This result is sharp only for $\lambda=0$, 1, 2. The special cases $\lambda=0$, 2 yield theorems proved earlier by Z. Nehari [Bull. Amer. Math. Soc. **55**, 545–551 (1949); these Rev. **10**, 696]. A number of lemmas on second order differential equations are also stated without proof.

A. W. Goodman (Lexington, Ky.).

Goluzin, G. M. On the theory of univalent functions. Mat. Sbornik N.S. 29(71), 197–208 (1951). (Russian)

Let S_r be the class of functions $f(z)=z+a_2z^2+\cdots$ univalent in $|z|<1$ and real on the real axis. Using a Loewner type differential equation for $f=f(z,t)$,

$$\frac{\partial f}{\partial t}=-f\frac{1-f^2}{1-2f\cos\theta(t)+f^2},\quad f(z,0)=z,\quad \lim_{t\to\infty}e^tf(z,t)=f(z),$$

where $f(z)$ belongs to a certain subclass of S_r; it is proved that if $f(z)\in S_r$, $\Im(z)>0$, $|z|<1$, then

$$|\arg f'(z)+\arg(1-z^2)|\leqq\begin{cases}2\arctan\dfrac{2\Im(z)}{1-|z|^2}, & \text{if } \arg\dfrac{1+z}{1-z}\leqq\dfrac{\pi}{4},\\[2ex] \dfrac{\pi}{2}+\log\dfrac{2\Im(z)}{1-|z|^2}, & \text{if } \arg\dfrac{1+z}{1-z}\geqq\dfrac{\pi}{4}.\end{cases}$$

From this inequality sharp upper and lower bounds for $\arg f'(z)$ are derived, and by a change of variable it is proved that if $g(t)$ is univalent in the strip $|\Im(t)|<1$, real on the real axis, $0<\Im(t)<1$, then

$$|\arg g'(t)|\leqq\begin{cases}\pi\Im(t), & \text{if } \Im(t)\leqq\frac{1}{2},\\ \frac{1}{2}\pi+\log\tan\frac{1}{2}\pi\Im(t), & \text{if } \Im(t)\geqq\frac{1}{2}.\end{cases}$$

The same methods yield sharp bounds for

$$|\log F'(\zeta)\pm\log(1-1/\zeta^2)|,$$

where $F(\zeta)=\zeta+a_0+a_1/\zeta+\cdots$ is univalent in $|\zeta|>1$ and real on the real axis.

Using the classic Loewner differential equation new proofs are given (1) of a theorem of Lebedev [Dissertation, Leningrad State University, 1951] and (2) of a theorem of Schiffer [Bull. Amer. Math. Soc. 54, 503–517 (1948); these Rev. 10, 26]. *A. W. Goodman* (Lexington, Ky.).

Goluzin, G. M. On majorants of subordinate analytic functions. I. Mat. Sbornik N.S. 29(71), 209–224 (1951). (Russian)

Let $f(z)$ and $F(z)$ be regular in $|z|<1$, $f(z)$ subordinate to $F(z)$, a_k and A_k the coefficients in the respective power series. The author proves a number of new and old inequalities, of which we quote the following. (3) If $\{\lambda_k\}$ is a nonincreasing nonnegative sequence, $\sum_{k=1}^n\lambda_k|a_k|^2=\sum_{k=1}^n\lambda_k|A_k|^2$. (4) For $0<r\leqq2-\frac{1}{2}$,

$$\int\int_{|z|<r}|f'(re^{i\theta})|^2rdrd\theta\leqq\int\int_{|z|<r}|F'(re^{i\theta})|^2rdrd\theta$$

(best possible constant). (5) For $0<r\leqq\frac{1}{2}$,

$$\int_0^{2\pi}|f'(re^{i\theta})|^2d\theta\leqq\int_0^{2\pi}|F'(re^{i\theta})|^2d\theta$$

(best possible constant). (6) For $p>0$ and $0<r\leqq2^{\frac{1}{2}}-1$, $\int_0^{2\pi}|f'(re^{i\theta})|^pd\theta\leqq\int_0^{2\pi}|F'(re^{i\theta})|^pd\theta$. (8) If $f(0)=F(0)=0$, $F(z)$ is univalent and $\arg f'(0)=\arg F'(0)$, then $|f(z)|\leqq|F(z)|$ for $|z|\leqq R$, where the best R is between .35 and $\frac{1}{2}(3-5^{\frac{1}{2}})$, the latter value being exact (9) when $F(z)$ maps $|z|<1$ on a region which is star-shaped with respect to $z=0$. (10) Under the hypotheses of (8), $|f'(z)|\leqq|F'(z)|$ for $|z|<.12$; the best constant does not exceed $3-8^{\frac{1}{2}}$. *R. P. Boas, Jr.*

Ullman, J. L. Two mapping properties of schlicht functions. Proc. Amer. Math. Soc. 2, 654–657 (1951).

Let D be the exterior of a simple analytic curve C and let $w=f(z)=z+a_0+a_1/z+\cdots$ map D one-to-one conformally onto $|w|>\rho$. The following two results are proved. (I) Let σ be a circle (centre z_0) whose closed interior lies in D. Then $f(z)/(z-z_0)$ maps σ onto a curve which is star-shaped from $w=0$. (II) If Z_1, Z_2 are two points of D such that each point of C is nearer to Z_1 than to Z_2 then $|f(Z_1)|<|f(Z_2)|$. Both proofs depend on the identity

$$\frac{f'(Z)}{f(Z)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{Z-z(\rho e^{i\alpha})},$$

where Z lies in D and $z(\rho e^{i\alpha})$ is the inverse image of $w=\rho e^{i\alpha}$ under $f(z)$. *W. K. Hayman* (Exeter).

Herzog, F., and Piranian, G. On the univalence of functions whose derivative has a positive real part. Proc. Amer. Math. Soc. 2, 625–633 (1951).

If $f(z)$ is regular and $\Re\{f'(z)\}>0$ in a convex domain D then $f(z)$ is univalent in D. For this property to hold of a given domain D it is shown in the present paper that it is necessary that the set of deficiency of D be totally disconnected and sufficient that D be almost convex. When D is of finite connectivity, it is necessary and sufficient that the set of deficiency is finite or null. It is not determined whether there can exist functions which are regular and of positive real part in an almost convex domain yet not analytically continuable into its convex hull. Such a continuation is established for a function bounded in a domain whose set of deficiency has zero perimeter. The geometrical significance of this classification is clarified by further theorems and examples. *A. Macintyre* (Aberdeen).

Heins, Maurice. A residue theorem for finite Blaschke products. Proc. Amer. Math. Soc. 2, 622–624 (1951).

Verf. beweist folgenden Satz: Besitzt ein endliches Blaschke-Produkt endliche Polstellen, so hat mindestens eine davon ein nicht verschwindendes Residuum. Daraus folgt, dass die Derivierten endlicher Blaschke-Produkte keine solche Produkte mehr sind. *A. Pfluger* (Zürich).

Ganin, M. P. On a general boundary problem for analytic functions. Doklady Akad. Nauk SSSR (N.S.) 79, 921–924 (1951). (Russian)

The author designates by S^+ a bounded $(\nu+1)$-connected region in the complex plane, bounded by $\nu+1$ simple, closed, "smooth" contours $L_0,\cdots,L_\nu$, without common points. Let U_α denote a class of functions such that: 1) every function of U_α transforms $L=L_0+\cdots+L_\nu$, one-to-one into itself, preserving directions; 2) every function of U_α has a first derivative nowhere zero, of a Hölder class on L. The set $S^-=S_0^-+\cdots+S_\nu^-$ is the complement of S^++L. By reducing to a system of integral equations in the sense of principal values, the author solves the following problem. To determine functions φ_p^+, φ_p^- $(p=1,\cdots,n)$, analytic in S^+, S^-, respectively, so that the boundary values on L satisfy the relations:

$$\sum_{p=1}^n\sum_{q=0}^{n_p}a_{kpq}(t)[\varphi_p^{(n_p-q)}(A)]^+ +\sum_{p=1}^n\sum_{q=0}^{m_p}b_{kpq}(t)[\varphi_p^{(m_p-q)}(B)]^- +\int_L\sum_{p=1}^n\sum_{q=0}^{n_p}c_{kpq}(t,\tau)[\varphi_p^{(n_p-q)}(C)]^+d\tau +\int_L\sum_{p=1}^n\sum_{q=0}^{m_p}d_{kpq}(t,\tau)[\varphi_p^{(m_p-q)}(D)]^-d\tau=g_k(t)$$

$\{k=1,\cdots,n$; the n_p, m_p certain integers; $A=\alpha_{p,n_p-q}(t)$, $B=\alpha^*_{p,m_p-q}(t)$, $C=\alpha_{p,n_p-q}(\tau)$, $D=\alpha^*_{p,m_p-q}(\tau)$ of class $U_\alpha\}$, the coefficients being of Hölder types and $\det\|a_{k_p0}\|\neq0$, $\det\|b_{k_p0}\|\neq0$. *W. J. Trjitzinsky* (Paris).

Tsuji, Masatsugu. A deformation theorem on conformal mapping. J. Math. Soc. Japan **2**, 213–215 (1951).

Let D be a simply connected domain on the z-plane, which contains $z=0$ and is contained in $|z|<M$. Let E be a continuum, which contains $z=0$ and is contained in D, such that a disc of radius ρ about any point of E is contained in D. If we map D conformally on $|w|<1$ by $w=w(z)$, $z=z(w)$ $(w(0)=0)$, then the image of E is contained in $|w|<1-k<1$, where $k=k(\rho/M)$ depends on ρ/M only (e.g., $k=(\rho/4M)\exp(-\alpha M^2/\rho^2)$, $\alpha=(64\pi/\sqrt{3})\log(32/9)$). A similar result can be obtained if D is a simply connected Riemann surface without branch points. *B. A. Amirà.*

Noshiro, Kiyoshi. Note on the cluster sets of analytic functions. J. Math. Soc. Japan **1**, 275–281 (1950).

Let D be an arbitrary simply connected region, C its boundary, E a closed subset of C of zero logarithmic capacity, z_0 any point of E, and $w=f(z)$ a single-valued meromorphic function in D. Cluster sets $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are defined as follows $S_{z_0}^{(D)}$ is the set of all values α for which $\lim_{\nu\to\infty} f(z_\nu)=\alpha$ for some sequence of points z_ν of D which tend to z_0. $S_{z_0}^{*(C)}$ is the intersection $\bigcap_\nu M_\nu$, where M_ν denotes the closure of the union $\bigcup_{\nu'} S_{z'}^{(D)}$ for all z' belonging to the intersection of $C-E$ and $|z-z_0|<r$. It follows that $S_{z_0}^{*(C)}\subset S_{z_0}^{(D)}$ and that $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is an open set [M. Tsuji, Proc. Imp. Acad. Tokyo **19**, 60–65 (1943); these Rev. **8**, 508]. The author proves that if Ω is not empty and Ω_a is any connected component of Ω, then $w=f(z)$ assumes infinitely often in every neighborhood of z_0 every value belonging to Ω_a, except at most two. This sharpens, for the case under consideration, a result of Kametani and Tsuji proved for general regions D [S. Kametani, Proc. Imp. Acad. Tokyo **17**, 429–433 (1941); these Rev. **7**, 380]. It is also shown that if, under the above conditions, $f(z)$ is assumed to be regular in the intersection of D with some neighborhood $U(z_0)$ of z_0, then the above conclusion holds with the two exceptional values reduced to one. Some consequences of these results are given. *W. Seidel* (Rochester, N. Y.).

Noshiro, Kiyoshi. A theorem on the cluster sets of pseudo-analytic functions. Nagoya Math. J. **1**, 83–89 (1950).

Let D be an arbitrary region in the plane of the complex variable $z=x+iy$ and let $f(z)$ be pseudo-meromorphic in D [for the definition of pseudo-meromorphic functions, see, for example, S. Kakutani, Jap. J. Math. **13**, 375–392 (1937)]. Let C be the boundary of D and E a closed subset of C of logarithmic capacity zero. Cluster sets $S_{z_0}^{(D)}$ and $S_{z_0}^{*(C)}$ are defined as in the preceding review. One of the results of the preceding paper is generalized as follows. If E is contained in a single boundary component C_0 of C, then $\Omega=S_{z_0}^{(D)}-S_{z_0}^{*(C)}$ is an open set. If, in addition, Ω is not empty, then $w=f(z)$ assumes infinitely often in every neighborhood of z_0 every value belonging to any connected component of Ω, except at most two values. *W. Seidel* (Rochester, N. Y.).

Pöschl, Klaus. Über die Wertverteilung der erzeugenden Funktionen Riemannscher Flächen mit endlich vielen periodischen Enden. Math. Ann. **123**, 79–95 (1951).

Riemann surfaces whose topological graph has p periodic ends were originally studied by E. Ullrich [Jber. Deutsch. Math. Verein. **46**, 232–274 (1936)] and later by H. Wittich [Arch. Math. **1**, 160–166 (1948); these Rev. **10**, 442]. The problem is to determine the order, defects and ramification indices of the corresponding meromorphic functions. The explicit determination is possible, due to the fact that the conformal mapping can be replaced by simpler quasi-conformal mappings. The order is

$$x=\frac{p}{2}+\frac{2}{p}\left(\frac{\log A}{2\pi}\right)^2$$

where A is a rational number which measures the asymmetry of the surface, and the defects and ramification indices turn out to be algebraic numbers in the field generated by $A^{1/p}$. *L. V. Ahlfors* (Cambridge, Mass.).

Röhrl, Helmut. Über Differentialsysteme, welche aus multiplikativen Klassen mit exponentiellen Singularitäten entspringen. I. Math. Ann. **123**, 53–75 (1951).

The class field theory of algebraic functions has inspired investigations in a similar style of classes of functions which are allowed to have multiplicative branch points with arbitrary (not necessarily rational) exponents. The present paper carries on the work of R. König and H. Schmidt, and introduces exponential singularities in addition to poles and multiplicative branch points. With given numbers a_ν, α_ν, $\nu=0, 1, \cdots, n$, and a given rational function $r(z)$, the class of functions considered in this memoir is generated by

$$W(z)=\prod_{\nu=0}^{n}(z-a_\nu)^{\alpha_\nu}e^{r(z)}.$$

The whole theory is modelled on H. Schmidt's theory of multiplicative functions and differential systems [Math. Ann. **105**, 325–380 (1931)]. The present first installment gives the arithmetical theory.

The class k consists of all functions $R(z)W(z)$, and the complementary class $\bar{k}$ of all functions $R(z)/W(z)$, where $R(z)$ is any rational function. Any subset of either of these classes is also called a class if it is closed under (i) addition, (ii) multiplication by a rational function of a certain class, (iii) differentiation. For instance, if $R(z)$ is restricted to rational functions which are regular except possibly at a finite set of (given) points, or to rational functions with coefficients in a certain field (which must contain, however, the a_ν, α_ν, and the coefficients of $r(z)$), we obtain such a class. The theory of divisors and multiples, and the definition of differentials, of elementary functions and differentials (whose principal parts have leading coefficient unity), the expansion and so-called interchange theorems, go through in the usual manner. The Riemann-Roch theorem is proved in two parts, one of which states the number of linearly independent functions, the other the number of linearly independent differentials, which are multiples of a fixed divisor. Reduction theorems lead to the construction of finite (linear) bases for classes whose singularities are contained in a fixed finite set of points.

In most of the paper the coefficient field is the field of complex numbers but in the reduction theorems the author assumes that the coefficients of $R(z)$ belong to any field, and proves the reduction theorems over a field obtained by adjoining the a_ν, α_ν, the coefficients and poles of $r(z)$, and the (fixed) set of (possible) poles of the $R(z)$ to the coefficient field. *A. Erdélyi* (Pasadena, Calif.).

Stein, Karl. Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem. Math. Ann. **123**, 201–222 (1951).

Ceci fait suite à un mémoire antérieur [Math. Ann. **117**, 727–757 (1941); ces Rev. **3**, 204]. Rappelons des définitions: une donnée de Cousin, dans un domaine X, consiste en la

donnée de fonctions f_i holomorphes dans des ouverts U_i recouvrant X, telles que, dans $U_i \cup U_j$, f_i/f_j soit holomorphe et $\neq 0$; le problème de Cousin consiste à trouver, si possible, une F holomorphe dans X telle que, dans chaque U_i, F/f_i soit holomorphe et $\neq 0$. Une fonction additivement automorphe dans X est une fonction F holomorphe dans le revêtement universel $\tilde{X}$ de X, et qui, par tout lacet de X, s'augmente d'une fonction holomorphe dans X; d'où un homomorphisme du groupe d'homologie (à coeff. entiers) $H_1(X)$ dans le groupe additif G des fonctions holomorphes dans X; cet homomorphisme est nul sur le sous-groupe de torsion $T_1(X)$ de $H_1(X)$, donc définit un homomorphisme du groupe de Betti $B_1(X) = H_1(X)/T_1(X)$ dans G; les images de cet homomorphisme s'appellent les périodes de F. Une fonction multiplicativement automorphe dans X est une fonction Φ, holomorphe dans $\tilde{X}$, qui, par tout lacet de X, est multipliée par une fonction holomorphe dans X et partout $\neq 0$; d'où un homomorphisme de $H_1(X)$ dans le groupe (multiplicatif) Γ des fonctions holomorphe dans X et à valeurs dans le groupe multiplicatif C^* des nombres complexes $\neq 0$; les images de cet homomorphisme s'appellent les périodes multiplicatives de Φ.

Voici les principaux résultats du mémoire; dans la suite, il est sous-entendu que X est un domaine d'holomorphie: (1) Pour tout homomorphisme $B_1(X) \to G$, il existe une fonction F, additivement automorphe dans X, admettant les périodes définies par cet homomorphisme (résultat annoncé indépendamment par H. Cartan, Confér. Congrès Intern. Math., Cambridge, Mass., 1950). (2) Pour qu'une fonction multiplicativement automorphe Φ puisse se mettre sous la forme $e^{2\pi i A}\Psi$, où A est additif automorphe et Ψ a pour périodes des racines de l'unité, il faut et il suffit que les périodes multiplicatives de Φ aient des logarithmes uniformes. (3) Pour qu'une donnée de Cousin admette une solution Φ, multiplicativement automorphe, dont les périodes soient des racines de l'unité, il faut et il suffit que l'intersection du $(2n-2)$-cycle (défini par la donnée de Cousin) avec tout 2-cycle entier soit nulle; la fonction Φ est alors déterminée à un facteur près, de la forme $e^{2\pi i A}$, où A est additivement automorphe avec périodes rationnelles. Pour que les périodes de Φ soient égales à un sur les éléments d'ordre fini de $H_1(X)$, il faut et il suffit que l'intersection du $(2n-2)$-cycle soit en outre nulle avec tout 2-cycle modulo p, et cela pour tout entier p. [Note du rapporteur: le mémoire analysé ne contient en réalité qu'une partie des résultats énoncés en (3).] (4) Etant donné arbitrairement un homomorphisme de $B_1(X)$ dans le groupe multiplicatif des racines de l'unité, il existe une fonction multiplicativement automorphe qui admet les périodes définies par cet homomorphisme.

Ces résultats sont en partie étendus au cas plus général où X, au lieu d'être un domaine d'holomorphie plongé dans l'espace numérique complexe, est une variété analytique-complexe satisfaisant à la condition: (α) l'espace X est "convexe" par rapport à la famille des fonctions holomorphes dans X, et il existe, en chaque point de X, un système de coordonnées locales formé de fonctions holomorphes dans tout X.

[Note du rapporteur: tous les résultats de l'auteur valent dans ce cas plus général. D'autre part, dans sa conférence citée, le rapporteur a montré qu'une donnée de Cousin définit un élément u du deuxième groupe de cohomologie à coeff. entiers $H^2(X)$; c'est l'invariant topologique d'un espace fibré de base X et de fibre C^*, défini par la donnée de Cousin; lorsque X est une variété analytique satisfaisant à la condition (α) ci-dessus, la nullité de u est nécessaire et suffisant pour que le problème de Cousin ait une solution uniforme. Dire que le $(2n-2)$-cycle défini par la donnée a une intersection nulle avec les 2-cycles entiers et les 2-cycles modulo p, revient à dire que u appartient au sous-groupe des éléments divisibles de $H^2(X)$, sous-groupe qui, d'après la théorie de l'homologie, s'identifie au groupe d'extensions Ext $(B_1(X), Z)$; on note Z le groupe additif des entiers. Le résultat (4) entraîne qu'il existe toujours une donnée de Cousin dont l'invariant u soit un élément arbitrairement donné de Ext $(B_1(X), Z)$.]

Le mémoire contient aussi quelques propositions d'algèbre; le résultat principal peut se formuler ainsi: soit B un groupe abélien dénombrable et sans torsion (par ex. le groupe de Betti $B_1(X)$ ci-dessus); alors ou bien B est libre, et par suite Ext $(B, Z) = 0$, ou bien Ext (B, Z) a une puissance supérieure au dénombrable (note: en fait, Ext (B, Z) a alors la puissance du continu). Enfin l'auteur donne, sans démonstration, l'intéressant résultat: dans l'espace à 2 dimensions complexes, il existe toujours un domaine d'holomorphie X dont le groupe de Betti $B_1(X)$ soit un groupe dénombrable, sans torsion, arbitrairement donné.

H. Cartan (Paris).

Hitotumatu, Sin. A condition of the domain of regularity. Kōdai Math. Sem. Rep. **1951**, 19–20 (1951).

Il s'agit du théorème connu: pour un domaine borné et univalent de l'espace de n variables complexes, il y a équivalence entre les propriétés suivantes: (a) D est domaine d'holomorphie; (b) D est convexe par rapport aux fonctions holomorphes dans D; (c) D est réunion d'une suite croissante de "polyèdres analytiques" relatifs à des fonctions holomorphes dans D. L'auteur donne une démonstration du fait que (c) implique (a). À noter que l'hypothèse que D est borné est inessentielle. *H. Cartan* (Paris).

Hitotumatu, Sin. Cousin problems for ideals and the domain of regularity. Kōdai Math. Sem. Rep. **1951**, 26–32 (1951).

Ceci est un raccourci de la théorie des idéaux de fonctions analytiques pour les domaines d'holomorphie, due à K. Oka et H. Cartan. La marche des démonstrations y est indiquée dans ses traits essentiels, mais les démonstrations difficiles des points cruciaux sont omises. Le seul résultat nouveau concerne une généralisation aux fonctions méromorphes d'un problème connu concernant les fonctions holomorphes. L'auteur se donne un idéal $\mathfrak{J}$ de fonctions holomorphes dans un ensemble E, engendré par un nombre fini de fonctions, et un recouvrement de E par des ouverts U_i dans chacun desquels on s'est donné des fonctions méromorphes f_i, telles que, dans $U_i \cap U_j$ supposé non vide, $f_i - f_j$ soit holomorphe et appartienne à l'idéal engendré par $\mathfrak{J}$; il s'agit de trouver une f méromorphe dans E, telle que, dans chaque U_i, $f - f_i$ soit holomorphe et appartienne à l'idéal engendré par $\mathfrak{J}$. Ce problème est résolu pour des "polyèdres analytiques" (compacts). [Note du rapporteur: la solution est la même que dans le cas où les f_i sont holomorphes; elle vaut donc aussi lorsque E est un domaine d'holomorphie général, sans qu'il soit besoin de supposer que l'idéal $\mathfrak{J}$ a un nombre fini de générateurs.]

L'article contient aussi une généralisation du théorème de la note analysé ci-dessus au cas où l'on se donne une famille de fonctions holomorphes dans D, contenant les constantes et les coordonnées complexes de l'espace ambiant, et qui permette d'approcher, uniformément sur tout compact, toute fonction holomorphe dans D. *H. Cartan.*

Rothstein, Wolfgang. Über die Fortsetzung vierdimensionaler analytischer Flächen. Arch. Math. 2 (1949–1950), 456–460 (1951).

A more detailed exposition of this paper appeared in Math. Ann. 121, 340–355 (1950); 122, 424–434 (1951); these Rev. 11, 652; 12, 818.

Fedorov, V. S. Monogenic vector-functions. I. Mat. Sbornik N.S. 29(71), 177–184 (1951). (Russian)

Let $\mathbf{r}$ be the radius vector drawn to a variable point M of a region D in 3-space, and let $\mathbf{f}(\mathbf{r})$ be a single-valued continuous vector-function of $\mathbf{r}$. Let $\Delta\mathbf{r}=\mathbf{m}\Delta s$ ($\mathbf{m}$ a unit vector, Δs a scalar) be an increment of $\mathbf{r}$, and let $\mathbf{f}_{\mathbf{m}}=\lim_{\Delta s\to 0}\Delta\mathbf{f}/\Delta s$ be the directional derivative of $\mathbf{f}$ at M. The function $\mathbf{f}$ is called A-monogenic in D if to every $M\in D$ there corresponds a closed set $A=A(\mathbf{f}, M)$ of unit direction vectors $\mathbf{m}$ such that (1) $\mathbf{f}_{\mathbf{m}}$ exists for every $\mathbf{m}\in A$ and depends continuously on $\mathbf{m}$ and on $M\in D$; (2) the scalar products $(\mathbf{f}_{\mathbf{m}})^2$ and $\mathbf{m}\cdot\mathbf{f}_{\mathbf{m}}$ are independent of $\mathbf{m}$; (3) $A(\mathbf{f}, M)$ contains, for each M, three coplanar but pairwise non-collinear unit vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, with $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=0$ (where α, β, γ are non-zero scalars) such that $\alpha\mathbf{f}_{\mathbf{a}}+\beta\mathbf{f}_{\mathbf{b}}+\gamma\mathbf{f}_{\mathbf{c}}=0$. If, for each M, $A(\mathbf{f}, M)$ contains all possible directions in 3-space, $\mathbf{f}$ is called completely A-monogenic. In the case that D is the cube $a'<x, y, z<a''$, it is proved that if $A(\mathbf{f}, M)$ contains every unit vector orthogonal to the z-axis, and if u, v, w are the projections of $\mathbf{f}$ on the x, y, z-axes respectively, then $u+iv=h(x+iy, z)$ and $w=F(z)$, in the sense that (1) for every fixed z, $u+iv$ is a holomorphic function of $x+iy$ in $a'<x, y<a''$; (2) w depends only on z in D. If, furthermore, $\mathbf{f}$ is completely monogenic in D, then $\mathbf{f}$ is a linear function of $\mathbf{r}$. These results are similar to those obtained by the author [Doklady Akad. Nauk SSSR (N.S.) 72, 637–639 (1950); these Rev. 12, 19] where a somewhat different definition of monogenicity is used.

A. J. Lohwater (Ann Arbor, Mich.).

Adauto da Justa Medeiros, Luiz. On monogenic areolar functions. Revista Cientifica 1, no. 3, 19–22 (1950). (Portuguese)

The author presents some special theorems in the theory of polygenic functions $f(z)=\phi(x, y)+i\psi(x, y)$ of the complex variable $z=x+iy$ [see Kasner and De Cicco, Univ. Nac. Tucumán. Revista A. 4, 7–45 (1944); these Rev. 7, 59]. In terms of the mean and phase derivatives $\partial/\partial z=\frac{1}{2}(\partial/\partial x-i\partial/\partial y)$ and $\partial/\partial\bar{z}=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$, a polygenic function of class C^2 is said to be areolar monogenic if $\partial^2 f/\partial\bar{z}^2=0$ in the given region. This class of functions was discussed by R. N. Haskel [Bull. Amer. Math. Soc. 52, 332–337 (1946); these Rev. 7, 381]. A monogenic function for which $\partial f/\partial\bar{z}=0$ is areolar monogenic. Moreover, the complete class of areolar monogenic functions is of the form $f(z)=\bar{z}a(z)+b(z)$, where $a(z)$ and $b(z)$ are both monogenic. A polygenic function of class C^4 for which $\nabla^2(\nabla^2 f)=\nabla^4 f=0$, where ∇^2 is the Laplace operator, is termed a biharmonic function. A biharmonic function is given by the fourth order partial differential equation $\partial^4 f/\partial z^2\partial\bar{z}^2=0$. If an areolar monogenic function is of class C^4, then it is biharmonic. Any biharmonic function is of the form $f(z)=h(z)+za(z)+\bar{z}b(z)$, where $h(z)$ is harmonic, $a(z)$ is reverse monogenic, and $b(z)$ is direct monogenic. All such functions are special analytic polygenic functions.

J. De Cicco (Chicago, Ill.).

Theory of Series

Agnew, Ralph Palmer. Ratio tests for convergence of series. Pacific J. Math. 1, 1–3 (1951).

Let the series $\sum a_n$ and $\sum b_n$ be related by

$$a_{n+1}/a_n=(1+c_n)(b_{n+1}/b_n). \tag{1}$$

It was proved by Jehle [Math. Z. 52, 60–61 (1949); these Rev. 11, 241] that either series converges when the other does, provided that the series $\sum c_n$ converges absolutely. A converse of this is proved in the present note. It is shown that in order that series $\sum a_n$ and $\sum b_n$ related by (1) should always converge or diverge together, it is necessary that $\sum c_n$ be absolutely convergent.

O. Frink.

Schmetterer, Leopold. Bemerkungen zur Multiplikation unendlicher Reihen. Math. Z. 54, 102–114 (1951).

The Laurent product $C=\sum_{-\infty}^{\infty}c_k$, where $c_k=\sum_{n+m=k}a_m b_n$, of two series, infinite in both directions, $A=\sum_{-\infty}^{\infty}a_k$, $B=\sum_{-\infty}^{\infty}b_k$ is discussed. Convergence of these series is in the "restricted" sense, as $\sum_{-n}^{n}a_k\to A$. The main results are Theorem 1. If $\sum a_k$ and $\sum b_k$ converge, and $a_k=O(|k|^{\delta-1})$, $b_k=o(|k|^{-\delta})$ where $0<\delta\leqq 1$, then $\sum c_k$ converges and $C=A\cdot B$. The theorem ceases to be true with two O-conditions. Theorem 2. If $0<\delta<1$, $a_k=O(|k|^{-\delta})$, $\sum b_k$ converges and $b_k=o(|k|^{\delta-\delta})$, then $\sum c_k$ is equiconvergent with $B\sum a_k$, i.e. $\sum(c_k-Ba_k)=0$. The theorem does not hold for $\delta=1$. The Fourier product of two series and the Laurent product of two double series $\sum a_{k,l}$ and $\sum b_{k,l}$, $-\infty<k, l<\infty$, is also discussed. [For previous work and similar results compare G. H. Hardy, Divergent series, Oxford, 1949, chapter X; these Rev. 11, 25.]

W. W. Rogosinski.

Austin, M. C. On limitation theorems for (A, λ) summability. J. London Math. Soc. 26, 304–307 (1951).

The author proves (essentially) that if the sequence $u_n\geqq 0$ is such that $\liminf(\lambda_n^{-1}\log u_n)\leqq 0$ and that the (A, λ) summability of $\sum a_n$ implies $a_n=O(u_n)$, then $\lambda_n(\Delta\lambda_n)^{-1}=O(u_n)$. As a corollary, an (A, λ) method is equivalent to convergence if it sums only series with bounded terms.

G. G. Lorentz (Toronto, Ont.).

Ryabcev, I. On the summability methods of S. N. Bernštein and Cesàro. Doklady Akad. Nauk SSSR (N.S.) 78, 869–872 (1951). (Russian)

It was shown by Karamata [Mat. Sbornik N.S. 21(63), 13–24 (1947); Math. Z. 52, 305–306 (1949); these Rev. 9, 140; 11, 347] that if $\sum u_n$ is a series with partial sums s_n, $h=\frac{1}{2}$,

$$B_n=\sum_{n=0}^{\infty}\left[\cos\frac{k\pi}{2(n+h)}\right]^n u_k, \quad B_n'=\frac{2s_0+2s_1+\cdots+2s_{n-1}+s_n}{2n+1},$$

then $\lim B_n=L$ if and only if $\lim B_n'=L$. The author attempts to show that, for each $r>1$, the Cesàro method C_r includes the above Bernstein method for evaluating series by proving the following proposition. If s_n is a sequence for which $\lim B_n'=L$, then s_n is evaluable C_r $(r>1)$ to L. [Errors in the proof appear in applications of the author's estimate (7) which is incorrect when α is a small positive number. We now show that the proposition is false by showing that C_r cannot include the above Bernstein method if $r<2$. If we put $M_n=(s_0+s_1+\cdots+s_n)/(n+1)$, then we can put B_n' in the form

$$B_n'=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)M_{n-1}+\frac{1}{2}\left(1+\frac{1}{2n+1}\right)M_n.$$

Let $0<h<1$ and let $M_n=(-1)^n n^h$. Then $\lim B_n'=0$. But $s_n=(n+1)M_n-nM_{n-1}=(-1)^n[(n+1)n^h+n(n-1)^h]$ so that $\lim (-1)^n s_n/n^{1+h}=2$. Therefore, since $\lim s_n/n^{1+h}$ must be zero if s_n is evaluable C_{1+h}, s_n is nonevaluable C_{1+h}. It was shown by Ogievecki\u{i} [Doklady Akad. Nauk SSSR (N.S.) 76, 635–638 (1951); these Rev. 12, 819] that C_2 does include the above Bernstein method]. *R. P. Agnew.*

***Heller, Isidore. Contribution à la théorie des séries divergentes.** Thesis, Université de Genève, Zürich, 1950. 20 pp.

The transformation $v_k=\sum_{n=1}^{\infty}c_{kn}u_n$ is such that $\sum v_n z^n$ has a positive radius of convergence whenever $\sum u_n z^n$ has a positive radius of convergence if and only if to each $\epsilon>0$ corresponds a number M_ϵ such that $|c_{kn}|\leqq\epsilon^n M_\epsilon^k$ for each $n, k=1, 2, 3, \cdots$. There is a treatment involving much more complicated conditions and with examples, of transformations $v_k=\sum c_{kn}u_n$ such that the series $\sum v_k z^k$ has a Mittag-Leffler star S containing the point $z=1$ and an analytic extension having the value σ at $z=1$ whenever the series $\sum u_n z^n$ has the same property. *R. P. Agnew.*

Teghem, J. Sur des transformations de séries. Application aux séries entières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 21–33 (1951).

The author discusses the applicability of the series-to-series transforms $P^{(-\alpha)}(\alpha)$ and $\Pi^{(\alpha)}(\alpha)$ [same Bull. (5) 36, 730–741 (1950); these Rev. 12, 695] to the summability of power series. If $\sum a_n z^n=f(z)$, convergent for $|z|<r$, then it is summable at most for points z with $|\alpha z|<r$, and if it is summable at z_1, it is absolutely summable to $f(z)$ (with a new value of α) at all points of a certain circle.

R. C. Buck (Madison, Wis.).

R.-Salinas, Baltasar. On a meromorphic function and its application to the summing of series. Gaceta Mat. (1) 3, 6–17 (1951). (Spanish)

The author obtains in closed form the sums of series of the form $\sum_0^\infty\phi(n)$ and $\sum_0^\infty\phi(n)e^{2\pi ni\theta}$ where $\phi(x)$ is a rational function, by means of residues, using the function

$$G(z)=-\gamma-\Gamma'(z)/\Gamma(z)=\frac{1}{z}+\sum_1^\infty\left(\frac{1}{z+v}-\frac{1}{v}\right)$$

which has simple poles at $0, -1, -2, \cdots$. *R. C. Buck.*

Popović, Božidar. Sur certaines théorèmes inverses de sommabilité de Cesàro. Srpska Akad. Nauka. Zbornik Radova, Knj. 7. Matematički Institut, Knj. 1, 83–90 (1951). (Serbo-Croatian. French summary)

Let k be a positive integer. Let $s(x)$ have bounded variation over each finite interval $0\leqq x\leqq x_0$, and let

$$\frac{k}{x}\int_0^x\left(1-\frac{t}{x}\right)^{k-1}s(t)dt=O(1)$$

as $x\to\infty$. Let $\varphi(x)$ be a monotone function for which $\varphi(x)\geqq 1$ and $\varphi(x+x\varphi(x)^{-1/(k+1)})\leqq N\varphi(x)$ as $x\to\infty$. Let $s(x)$ satisfy, for $\epsilon>0$, the Tauberian condition (of Robert Schmidt type) $s(x')-s(x)\geqq -w(\epsilon)$ whenever

$$x\leqq x'\leqq x+\epsilon x/\phi(x).$$

Then $s(x)=O([\varphi(x)]^{k/(k+1)})$ as $x\to\infty$. *R. P. Agnew.*

Korevaar, Jacob. An estimate of the error in Tauberian theorems for power series. Duke Math. J. 18, 723–734 (1951).

Let $f(t)=\sum_{n=0}^\infty a_n e^{-nt}$ $(t>0)$, $a_n>-n^{\alpha-1}L(n)$ $(n\geqq 1)$, where $\alpha\geqq 0$ and $L(u)$ is a slowly oscillating function (i.e. a continuous function such that $L(au)/L(u)\to 1$ as $u\to\infty$, for every $a>0$). Let $\rho(t)$, $\rho^*(n)$ be defined by

$$f(t)=t^{-\alpha}L(t^{-1})\{A+\rho(t)\},$$

$$s_n=\sum_{\nu=0}^n a_\nu=n^\alpha L(n)\left(\frac{A}{\Gamma(\alpha+1)}+\rho^*(n)\right).$$

Hardy and Littlewood proved essentially that, if $\rho(t)\to 0$ as $t\downarrow 0$, then $\rho^*(n)\to 0$ as $n\to\infty$. This paper considers the problem of making inferences from the order of $\rho(t)$ to that of $\rho^*(n)$, and extends the discussion to Cesàro sums $s_n^{(k)}$ in place of s_n. It is proved, in particular, that, if $L(u)\equiv 1$ and $\rho(t)=O(t)$, then $\rho^*(n)=O\{(\log n)^{-1}\log\log n\}$. It is shown that this estimate certainly cannot be improved beyond $O\{(\log n)^{-1}\}$.

The proof of the positive results is based on Karamata's method refined by the use of theorems on best approximation by polynomials. But it is stated that this method alone (or Littlewood's method of repeated differentiation) would give only $(\log n)^{-\frac{1}{2}}$ instead of $(\log n)^{-1}$. [In the case $\alpha=1$, $L(u)\equiv 1$, $a_n\geqq 0$, it has already been proved recently by the former method that $\rho(t)=O(t)$ implies $\rho^*(n)=O\{(\log n)^{-\frac{1}{2}}\}$; see Postnikov, Doklady Akad. Nauk SSSR (N.S.) 77, 193–196 (1951); these Rev. 12, 820.] The improvement is effected by applying this idea to obtain a preliminary estimate of a Cesàro sum of high order, and then arguing down from this by a Tauberian theorem of known type for Cesàro summability. *A. E. Ingham.*

Szász, Otto. On a Tauberian theorem for Abel summability. Pacific J. Math. 1, 117–125 (1951).

Extending earlier results of his own and a more recent one of A. Rényi [Acta Univ. Szeged. Sect. Sci. Math. 11, 119–123 (1946), these Rev. 8, 147] the author shows that Abel summability of $\sum a_n$ (a_n real) combined with any one of the following conditions implies convergence. Whenever $n\to\infty$ and $m/n\to 1$, (1) $V_n=\sum_{\nu\leqq n\nu}|a_\nu|=O(n)$ and $m^{-1}V_m-n^{-1}V_n\to 0$; (2) $U_n=\sum_{\nu\leqq n\nu}(|a_\nu|-a_\nu)=O(n)$ and $m^{-1}U_m-n^{-1}U_n\to 0$; (3) $n^{-1}(U_m-U_n)\to 0$;

(4) $$\sum_{n<\nu\leqq m}(|a_\nu|-a_\nu)\to 0.$$

Conditions (2), (3) and (4) are equivalent; it is shown that (1) is implied by Rényi's condition $n^{-1}V_n\to\lambda$, and that (1) implies (2). The first part of (2) implies that $\sum a_n$, being Abel summable, is $(C, 1)$-summable; the whole of (2) then implies convergence. *J. Korevaar* (Delft).

Slater, L. J. A new proof of Rogers's transformations of infinite series. Proc. London Math. Soc. (2) 53, 460–475 (1951).

The transformations of Rogers are of the general type $a_1m_1+a_2m_2+\cdots=b_1n_1+b_2n_2+\cdots$, where the m_r depend on the n_r, the b_r on the a_r [Proc. London Math. Soc. (1) 24, 337–352 (1893); (1) 25, 318–343 (1894); (1) 26, 15–32 (1895)]. His results, listed in groups A–G, and a condensed version of his methods, were given in a later paper [ibid. (2) 16, 315–336 (1917)]. Typical results are

$$a_0+a_2+a_4+\cdots=b_0+b_2q-b_4q^3+b_6(q^5+q^7)-b_8q^{12}-b_{10}q^{16}+\cdots,$$

$$a_1+a_3+a_5+\cdots=b_1-b_3(q+q^2)+b_5q^5+b_7q^7-b_9(q^{12}+q^{15})+\cdots,$$

where

$$a_{2n}=\frac{(q^{\frac{1}{2}}/u)_n(q^{\frac{1}{2}}/v)_n}{(q)_{2n}}u^nv^n,$$

$$a_{2n+1}=\frac{(q/u)_n(q/v)_n}{(q^2)_{2n}}u^nv^n,$$

$$b_{2n}=\Pi\left(\begin{matrix}uq^{\frac{1}{2}}, vq^{\frac{1}{2}}; q\\ q, uv;\end{matrix}\right)\frac{(q^{\frac{1}{2}}/u)_n(q^{\frac{1}{2}}/v)_n}{(uq^{\frac{1}{2}})_n(vq^{\frac{1}{2}})_n}u^nv^n,$$

$$b_{2n+1}=\Pi\left(\begin{matrix}uq, vq; q\\ q^2, uv;\end{matrix}\right)\frac{(q/u)_n(q/v)_n}{(uq)_n(vq)_n}u^nv^n,$$

and

$$(a)_0=1,\quad (a)_n=(a)_{q,n}=(1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$\Pi\left(\begin{matrix}a_1, \cdots, a_r; q\\ b_1, \cdots, b_s;\end{matrix}\right)=\prod_{n=0}^{\infty}\frac{(1-a_1q^n)\cdots(1-a_rq^n)}{(1-b_1q^n)\cdots(1-b_sq^n)}.$$

Bailey [ibid. (2) **49**, 421–435 (1947); **50**, 1–10 (1948); these Rev. **9**, 263, 585] has simplified Rogers' work and given a general summation formula

$$(1)\quad \sum_{n=0}^{\infty}(y)_n(z)_n\left(\frac{x}{yz}\right)^n\beta_n$$
$$=\Pi\left(\begin{matrix}x/y, x/z; q\\ x, x/yz;\end{matrix}\right)\sum_{n=0}^{\infty}\frac{(y)_n(z)_n}{(x/y)_n(x/z)_n}\left(\frac{x}{yz}\right)^n\alpha_n,$$

where $\alpha_0=1$ and

$$(2)\qquad \beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(x)_{n+r}(q)_{n-r}}.$$

At Bailey's suggestion, the author has used (1) to derive all of Rogers' results and many new ones. To evaluate (2) in special cases, he has used another summation formula of Bailey's [Generalized hypergeometric series, Cambridge Tract, no. 32, 1935, 8.3 (1)]. A typical new result is

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_{2n}}=\prod_{n=1}^{\infty}(1+q^{10n-16})(1+q^{10n-14})(1-q^{10n})$$
$$-q^2\prod_{n=1}^{\infty}(1+q^{10n-28})(1+q^{10n-4})(1-q^{10n}).$$

N. J. Fine (Philadelphia, Pa.).

Karadžić, Lazar. Un théorème sur des séries fonctionelles. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **5**, 145–161 (1950). (Serbo-Croatian. French summary)

Some elementary considerations relating the uniform convergence on $[a, b]$ of $\sum u_i(x)v_i(x)$ to that of series of the form $\sum\Delta^n u_0(x)F^{(n)}(1)/n!$, where $F(z)$ is regular at the point 1, and where $\{u_i(x)\}$ is arbitrary provided $\Delta^n u_0(x)$ converges uniformly on $[a, b]$. Applications, all of them well known, are mentioned. *S. Mandelbrojt* (Houston, Tex.).

Isaacs, G. L. On a theorem due to M. Riesz. J. London Math. Soc. **26**, 285–290 (1951).

A proof of the following statement of M. Riesz [Acta Univ. Szeged. Sect. Sci. Math. **2**, 18–31 (1924)]. If $C(u)$ is of bounded variation in every finite interval $[0, \omega]$ and $\int_0^\omega e^{-us}dC(u)$ is summable (C, k) as $\omega\to\infty$ $(k\geqq 0, \Re s=\sigma>0)$, then $(1/\Gamma(k))\int_0^\omega(\omega-t)^{k-1}C(t)dt=o(e^{\sigma\omega}\omega^k)$.

W. H. J. Fuchs (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Sunouchi, Gen-ichirô. Notes on Fourier analysis. XLIV. On the summation of Fourier series. Tôhoku Math. J. (2) **3**, 114–122 (1951).

It is well known that the partial sums of a Fourier series at a point of continuity of the function are $o(\log n)$, and that this is a best possible result [G. H. Hardy, Proc. London Math. Soc. (2) **12**, 365–372 (1913); see E. C. Titchmarsh, The theory of functions, Oxford, 1932, p. 418]. Here the author proves (1) if $\varphi_\alpha(t)=o\{[\log(1/t)]^r\}$, $\alpha>0$, $r>1$, then $s_n^\alpha=o\{[\log n]^{r+1}\}$, and this is a best possible result, where $\varphi_\alpha(t)$, s_n^α are respectively the Cesàro means of order α of $\frac{1}{2}\{f(x+t)+f(x-t)\}$ and of the partial sums of the Fourier series at $t=x$. When $-1<r\leqq 1$, the Fourier series is also summable (C), and a Tauberian lemma of Zygmund yields (2) if $\varphi_\alpha(t)=o\{[\log(1/t)]^{\alpha-1}\}$, $0<\alpha\leqq 1$, then the Fourier series is summable $(R, \log n, \alpha)$. This supplements a result of F. T. Wang [Duke Math. J. **15**, 5–10 (1948); these Rev. **9**, 580] that the Fourier series is summable $(R, \log n, \alpha)$, $\alpha\geqq 1$, if $\varphi_\alpha(t)=o(1)$. For $\alpha=2, 3, \cdots$, the author gives the result (3) if $\varphi_\alpha(t)=o(1)|C, 0|$, then the Fourier series is summable $|R, \log n, \alpha|$. The proof is based on a Tauberian lemma which is proved, but stated incorrectly [$O\{(\log N)^\alpha\}$ should be replaced by $O(\log N)$]. The lemma could probably be replaced by the more general result: if $\sum a_n$ is summable $|C|$ and $s_n^\alpha/(\log n)^\alpha=o(1)|C, 0|$, then $\sum a_n$ is summable $|R, \log n, \alpha|$; this is strictly analogous to Zygmund's lemma.

L. S. Bosanquet (London).

Matsuyama, Noboru. Notes on Fourier analysis. XL. On the absolute summability of the Fourier series. Tôhoku Math. J. (2) **3**, 39–44 (1951).

Extending a result given incorrectly by Izumi and Kawata [Tôhoku Math. J. (1) **45**, 134–144 (1938)] and completed by Izumi and the author [Math. Japonicae **1**, 140–150 (1949); these Rev. **11**, 657], the author now proves that if $\varphi(t)=O\{(\log(k/t)^{-\beta}\}$, $\beta>0$, in $(0, \pi)$, then the Fourier series of the even function $\varphi(t)$ is summable $|R, \exp(\log n)^\alpha, 1|$, where $0<\alpha<\min(\beta, 1)$. He also gives a result which is included in the following theorem obtained recently by Mohanty [Proc. London Math. Soc. (2) **52**, 295–320 (1951); see also Mohanty, ibid. **51**, 186–196 (1949); these Rev. **12**, 822; **11**, 99]: if $\varphi(t)\log\log(k/t)$ is of bounded variation in $(0, \pi)$, then the Fourier series is summable $|R, \exp(\log n)^\alpha, 1|$ for $\alpha>1$. *L. S. Bosanquet* (London).

Sinvhal, S. D. On Cesàro non-summability of Fourier series. III. Ganita **1**, 27–30 (1950).

F. T. Wang [J. London Math. Soc. **22**, 40–47 (1947); these Rev. **9**, 182] proved that the Fourier series of an even function $\varphi(t)$ is summable $(C, 1)$ at the origin if $\varphi(t)=o\{(\log(1/t))^{-1}\}$ $(C, 1)$ as $t\to 0$. Here the author proves that the Fourier series need not be summable $(C, 1)$ if $\varphi(t)=o\{(\log(1/t))^{-\alpha}\}$, $0<\alpha<1$, nor indeed if

$$\int_0^t u^{-1}\varphi(u)du=o\{(\log(1/t))^{-\alpha}\}.$$

The author does not quote Wang's result, but a statement by F. C. Hsiang [Acad. Sinica Science Record **2**, 46–54 (1947); these Rev. **9**, 235] that the Fourier series is summable $(C, 1)$ if $\varphi(t)=O\{(\log(1/t))^{-1}\}$. It must, however, be pointed out that S. Izumi and G. Sunouchi have recently given an example to show that o cannot be replaced by O in Wang's result [Izumi and Sunouchi, Tôhoku Math. J.

(2) 1, 313–326 (1950); these Rev. 12, 405, 1003 (the review stated the contrary!)]. Their example is $\varphi(t)\sum\psi_\nu(t)$ where $\psi_\nu(t)=(-1)^\nu\alpha_\nu(d/dt)(t\sin M_\nu t)$ in $[\pi n_\nu^{-1}, \pi(n_\nu^{-1}+m_\nu^{-1})]$ and $=0$ otherwise, the α_ν, M_ν, n_ν, m_ν being strictly chosen. The reviewer agrees with their conclusion, but would prefer the parameters to be chosen so that $\alpha_\nu=(\log\mu_\nu)^{-1}$, $M_\nu=\mu_\nu^3$, $n_\nu=\mu_\nu^3$, $m_\nu=\mu_\nu^2$ where μ_ν is a rapidly increasing sequence of integers; this is to provide for the convergence of $\sum\alpha_\nu M_\nu m_\nu^{-2}$ which seems to be required. *L. S. Bosanquet.*

Prasad, B. N., and Siddiqi, J. A. On the Nörlund summability of the rth derived Fourier series. J. Indian Math. Soc. (N.S.) 14, 159–170 (1950).

A. Zygmund [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. 1925, 207–217] proved that, if the generalized derivative $f^{(r)}(x)$ exists, the rth derived series of $f(x)$ is summable (C, k), $k>r$, to the value $f^{(r)}(x)$. This theorem was generalized for Nörlund summation by Astrachan [Duke Math. J. 2, 543–568 (1936)]. The authors give a generalization of the last theorem analogous to the one given by F. T. Wang [Tôhoku Math. J. (1) 39, 399–405 (1934)]. *S. Izumi* (Tokyo).

Matsuyama, Noboru. Notes on Fourier analysis. XXXVII. On the convergence factor of the Fourier series at a point. Tôhoku Math. J. (2) 2, 126–134 (1950).

Wang proved that if $\{f(x+t)-f(x-t)\}=O(t^r)$ (C, α) as $t\to 0$, $r\to -1$, then $\sum\lambda_n(a_n\cos nx+b_n\sin nx)$ converges for $\lambda_n=n^{-\alpha/(\alpha+1+r)}$ [F. T. Wang, Sci. Rep. Tôhoku Univ., Ser. 1. 24, 665–696 (1936); see also S. Izumi and G. Sunouchi, Tôhoku Math. J. (2) 1, 313–326 (1950); these Rev. 12, 405, 1003]. Here the author quotes the case $\alpha=1$, $r>0$, states as a theorem the case $\alpha=1$, $r>0$, and shows that the result in these cases is best possible. He also states that a result of converse type due to Wang is best possible, but replaces this statement by a weaker result. *L. S. Bosanquet.*

Bari, N. K. The uniqueness problem of the representation of functions by trigonometric series. Amer. Math. Soc. Translation no. 52, 90 pp. (1951).

Translated from Uspehi Matem. Nauk (N.S.) 4, no. 3(31), 3–68 (1949); these Rev. 11, 26.

Bugaec, P. T. An asymptotic estimate of the remainder in the approximation of functions of two variables by Fourier sums. Doklady Akad. Nauk SSSR (N.S.) 79, 557–560 (1951). (Russian)

Let H_{ω_1,ω_2} be the class of functions $f(x, y)$ of period 2π in both x and y and satisfying the condition

$$|f(x_2, y_2)-f(x_1, y_1)|\leqq\omega_1(|x_2-x_1|)+\omega_2(|y_2-y_1|).$$

Extending the corresponding result obtained by S. M. Nikolsky [C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 191–194 (1946); these Rev. 8, 149] for functions of a single variable, the author obtains for

$$E_{mn}=\sup_{f\in H_{\omega_1,\omega_2}}|S_{mn}(f; x, y)-f(x, y)|$$

(S_{mn} is the partial sum of the Fourier series of f) the following asymptotic formula:

$$E_{mn}=2\pi^{-4}\theta_{mn}(2m+1)(2n+1)\log m\log n\times\int_0^{\frac{1}{2}\pi/(m)}\int_0^{\frac{1}{2}\pi/(n)}\min\{\omega_1(2u), \omega_2(2v)\}\times\sin(m+\tfrac{1}{2})u\sin(n+\tfrac{1}{2})v\,du\,dv+\rho_{mn}$$

where $h^{(m)}=2\pi/(2m+1)$,

$$\rho_{mn}=O\{(\log m+\log n)(\omega_1(h^{(m)})+\omega_2(h^{(n)}))\}$$

and $\frac{1}{2}\leqq\theta_{mn}\leqq 1$. If ω_1 and ω_2 are convex, then $\theta_{mn}=1$.

A. Zygmund (Chicago, Ill.).

Matveev, I. V. On methods of summation of double Fourier series for functions of two variables. Mat. Sbornik N.S. 29(71), 185–196 (1951). (Russian)

Let $f(x, y)$ be any continuous function of period 2π with respect to x and y, and let $\sum c_{kl}e^{i(kx+ly)}$ be the Fourier series of f. The author investigates the problem of conditions for the multipliers λ_{kl}^{mn} if the linear means

$$\sum_{k=-m}^{m}\sum_{l=-n}^{+n}c_{kl}e^{i(kx+ly)}\lambda_{kl}^{mn}\qquad(\lambda_{\pm k,\pm l}=\lambda_{k,l})$$

are to tend to $f(x, y)$, uniformly in (x, y), and for every continuous f. He obtains a set of necessary conditions and another set of sufficient conditions, both of which are not simple enough to be reproduced here. If the multipliers λ_{kl}^{mn} as functions of k and l are regular enough, the necessary conditions are also sufficient. The corresponding ideas for Fourier series of functions of a single variable had been developed by S. M. Nikol'skiĭ [Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 259–278 (1948); these Rev. 10, 247] and B. Sz.-Nagy [Acta Sci. Math. Szeged 12B, 204–210 (1950); these Rev. 11, 656]. *A. Zygmund* (Chicago, Ill.).

Gál, István Sándor. Sur la convergence d'interpolations linéaires. II. Corrections et améliorations concernant le cas des fonctions bornées. C. R. Acad. Sci. Paris 233, 347–350 (1951).

[For the first part of the paper, see the same C. R. 230, 1374–1376 (1950); these Rev. 11, 659.] The author investigates the same type of problem as in the first note. More general formulations are obtained, and comments are made about the dependence of conditions considered in that note.

A. Zygmund (Chicago, Ill.).

Stečkin, S. B. On absolute convergence of orthogonal series. I. Mat. Sbornik N.S. 29(71), 225–232 (1951). (Russian)

Let $f(x)$ be a continuous function of period 2π, $E_n(f)$ its best approximation by trigonometric polynomials of order $n-1$, $\omega(\delta, f)$ its modulus of continuity. S. Bernstein proved that the Fourier series of $f(x)$ converges absolutely if either $\sum n^{-\frac{1}{2}}E_n(f)$ or $\sum n^{-\frac{1}{2}}\omega(n^{-1}, f)$ converges [C. R. Acad. Sci. Paris 199, 397–400 (1934)]. The author shows first that these two conditions are equivalent. He then extends the first theorem (and its generalizations) to more general orthogonal series. Let $\{n_k\}$ be an increasing sequence of integers, $F(u)$ an increasing concave function vanishing at 0. Theorem 2. If $f\in L^2$ and c_n are its Fourier coefficients with respect to the orthonormal system $\Phi=\{\phi_n(x)\}_{n=1}^{\infty}$, then

$$\sum F(c_{n_k}^2)\leqq 2\sum F(k^{-1}\{E_{n_k}^{(2)}[f, \Phi]\}^2),$$

where the expression in braces is the mean-square difference between $f(x)$ and the sum of the first $n-1$ terms of its expansion in terms of Φ. Theorem 3.

$$\sum|c_{n_k}|\leqq(4/3)^{\frac{1}{2}}\sum k^{-\frac{1}{2}}E_{n_k}^{(2)}[f, \Phi].$$

Let $\omega^{(2)}(\delta, f)$ be the mean-square modulus of continuity of f, and a_n, b_n its ordinary Fourier coefficients. Theorem 4. $\sum(|a_{n_k}|+|b_{n_k}|)\leqq C\sum k^{-\frac{1}{2}}\omega^{(2)}(n_k^{-1}, f)$. Several theorems on absolute convergence of Fourier series follow as corollaries.

The proofs of Theorems 3 and 4 depend on the following inequality: if $u_n \geqq 0$, $u_n \neq 0$, $\sum u_n^2 < \infty$, then

$$\sum_{n=1}^{\infty} u_n < (4/3)^{\frac{1}{2}} \sum_{n=1}^{\infty} \{n^{-1}(u_n^2 + u_{n+1}^2 + \cdots)\}^{\frac{1}{2}}.$$

R. P. Boas, Jr. (Evanston, Ill.).

Beurling, Arne. On a closure problem. Ark. Mat. 1, 301–303 (1951).

Soit $f(x) \in L^p(-\infty, +\infty)$ pour tout p, $1 \leqq p < \infty$. Si dans un L^p, le sous-espace vectoriel V engendré par les translatées de f n'est pas dense, il en est de même dans L^r, pour $r \leqq p$, et il existe une fonction $g \in L^q$, $(1/p)+(1/q)=1$, telle que $f * g = 0$; on peut supposer g continue et bornée. On en tire (inégalité de Hölder), si $p \leqq 2$, que $\int_{-\infty}^{+\infty} |g(x)|^2 |x|^{\beta-1} dx < \infty$, dès que $\beta < 2/q = 2-(2/p)$. Soient F (fonction continue) et G (distribution) les transformées de Fourier de f et g. D'après les théorèmes taubériens de Wiener-Beurling, correspondant à $p=1$, $q=\infty$, on sait que le support de G (spectre de g) est contenu dans l'ensemble E des zéros de F. Mais G est une distribution d'énergie finie dans la théorie du potentiel de noyau $|x|^{-\beta}$, donc la β-mesure spectrale [Beurling, Analyse harmonique, Colloques Internationaux . . . , no. 15, pp. 9–29, Paris, 1949; ces Rev. 11, 429; voir aussi Deny, Acta Math. 82, 107–183 (1950); ces Rev. 12, 98] du support de G est >0, et a fortiori celle de E, donc la dimension de Hausdorff de E, α, est $\geqq \beta$. D'où la conclusion $\alpha \geqq 2-(2/p)$, ou $p \leqq 2/(2-\alpha)$. Le sous-espace V est donc dense dans L^p dès que $\alpha < 1$ et $p > 2/(2-\alpha)$. Un théorème démontré dans un autre article par Salem [voir l'analyse ci-dessous] montre que la constante $2/(2-\alpha)$ ne peut pas être améliorée. [Note du reviewer: La même démonstration est valable dans l'espace euclidien R^n, l'inégalité étant remplacée par $p \leqq 2n/(2n-\alpha)$, pourvu toutefois que $\alpha > n-2$.]

L. Schwartz (Nancy).

Salem, R. On singular monotonic functions whose spectrum has a given Hausdorff dimension. Ark. Mat. 1, 353–365 (1951).

Quel que soit α, $0 < \alpha \leqq 1$, il existe une mesure singulière $\mu \geqq 0$, dont le support est un compact de la droite réelle, de mesure nulle, de dimension de Hausdorff α, dont la transformée de Fourier $\gamma(u)$ soit dans L^q pour tout $q > 2/\alpha$. La constante $2/\alpha$ ne peut pas être améliorée. On peut aussi faire en sorte que, pour n entier, $|\gamma(n)| \leqq \Omega(n)/|n|^{\alpha/2}$, $\Omega(n)$ croissant plus lentement, pour $|n| \to \infty$, que toute puissance >0 de n. La méthode est la suivante: par des procédés de subdivision analogues au procédé classique de Cantor-Lebesgue, on forme des mesures μ_t, dépendant d'un paramètre t, $0 \leqq t \leqq 1$, et on montre que $\int_{-\infty}^{+\infty} du \int_0^1 |\gamma_t(u)|^q dt < \infty$, ce qui, d'après le théorème de Fubini après interversion des intégrations, montre que $\gamma_t(u)$ est dans L^q pour presque toutes les valeurs de t. D'où la conclusion. Cette méthode ne donne pas explicitement une mesure μ ayant les propriétés indiquées, ce qui est sans importance pour le résultat cherché ici. [Note du reviewer: On peut ensuite prouver qu'on peut trouver μ telle que $|\gamma(u)| \leqq \Omega(u)/|u|^{\alpha/2}$ pour $|u|$ (non entier) $\to \infty$.]

L. Schwartz (Nancy).

Edwards, R. E. On derivative and translational bases for periodic functions. Proc. Amer. Math. Soc. 2, 644–653 (1951).

Soit E l'espace de Banach des fonctions continues sur le groupe additif G des nombres réels (mod 1), E' son dual topologique. Pour que le sous-espace vectoriel fermé $D(f)$ engendré par les dérivées de $f \in E$, supposée indéfiniment dérivable, coïncide avec le sous-espace vectoriel fermé $T(f)$ engendré par ses translatées f_t, $t \in G$, il faut et il suffit que la classe $C(f)$ des fonctions $f * \mu$, $\mu \in E'$, soit quasi-analytique Δ (c.à.d. que toute fonction de cette classe, ayant toutes ses dérivées nulles en un point, soit identiquement nulle). Pour que le sous-espace vectoriel fermé $T(A, f)$ engendré par les translatées f_t, $t \in A$, soit identique à $T(f)$ quel que soit l'ouvert non vide A de G, il faut et il suffit que $C(f)$ soit quasi-analytique I (c.à.d. que toute fonction de cette classe, nulle sur un intervalle, soit identiquement nulle). Une condition suffisante pour que $D(f) = T(f)$ est que les coefficients de Fourier $F(n)$ de f admettent une majoration $|F(n)| \leqq \exp(-p(|n|))$, $p(t)$ étant une fonction $\geqq 0$ dérivable de $t \geqq 0$, $tp'(t)$ étant croissante et tendant vers ∞ pour $t \to \infty$, l'intégrale $\int_1^{\infty} p(t) t^{-2} dt$ étant infinie. Une condition suffisante pour que $T(A, f) = T(f)$ pour tout A ouvert non vide de G, est que les coefficients de Fourier $F(n)$ admettent la même majoration, sauf peut-être pour une suite n_i de valeurs de n telle que $\sum |1/n_i| < \infty$. Ces résultats sont très voisins de ceux de Mandelbrojt [C. R. Acad. Sci. Paris 231, 16–18 (1950); ces Rev. 12, 32]. Les translatées f_t de f sont dites indépendantes si, quels que soient A fermé $\subset G$, et t_0 non-$\in A$, l'ensemble $T(A, f)$ ne contient pas f_{t_0}. Une condition nécessaire et suffisante d'indépendance des translatées de f est que la classe $C(f)$ contienne des fonctions $\neq 0$ s'annulant en dehors de voisinages arbitraires de l'unité de G. Une condition suffisante est la suivante: $F(n) \neq 0$ pour tout n, et $\sum_{n \neq 0} |F(n)|^{-1} A(s(|n|))/|n|^{s(|n|)} < +\infty$, où $s(p)$ est une suite convenable d'entiers $\geqq 0$, $A(s)$ une fonction $\geqq 0$ telle que, si $B(s) = (A(s))^{1/s}$, $\bar{B}(s) = \inf_{t \geqq s} B(t)$, on ait $\sum_{i=1}^{\infty} 1/\bar{B}(s) < +\infty$. Cette condition sera améliorée dans un mémoire ultérieur.

L. Schwartz (Nancy).

Burkill, H. Almost periodicity and nonabsolutely integrable functions. Proc. London Math. Soc. (2) 53, 32–42 (1951).

La distance entre deux fonctions $f(x)$ et $g(x)$ étant définie par

$$\sup_{\substack{0 \leqq h \leqq 1 \\ -\infty < x < \infty}} \left| (D) \int_x^{x+h} [f(t) - g(t)] dt \right|,$$

où (D) indique que l'intégrale est à prendre au sens de Denjoy; la notion de presque-périodicité peut, à partir de là, s'étendre de la manière habituelle aux fonctions intégrables au sens de Denjoy. L'auteur démontre alors les théorèmes sur la somme, l'existence de la valeur moyenne, l'existence de la série de Fourier, et le théorème d'unicité. Il établit que la fermeture D de l'ensemble des polynomes trigonométriques coïncide avec la classe des fonctions presque-périodiques qui vient d'être définie, puis généralise les résultats de Pollard [Proc. London. Math. Soc. (2) 27, 209–222 (1927)] sur la sommabilité des séries de Denjoy-Fourier.

J. Favard (Paris).

Whitcraft, William A., Jr. A series expansion of the Fourier integral. Proc. I. R. E. 39, 1087–1089 (1951).

Doetsch, Gustav. Handbuch der Laplace-Transformation. Band I. Theorie der Laplace-Transformation. Verlag Birkhäuser, Basel, 1950. 581 pp. 74 Swiss francs; bound, 78 Swiss francs.

This is the first volume of a two volume treatise on the Laplace transform (unilateral and bilateral). This volume is devoted to theory, while the second volume will deal with

applications. The scope of the present volume is quite extensive and it offers a satisfactory survey of the field. Here, as in the author's "Theorie und Anwendung der Laplace-Transform" [Springer, Berlin, 1937], the emphasis is upon those aspects of the theory which are needed in the problems of engineering and physics. In addition to the large amount of material developed in detail, certain subjects which are fully treated in other readily available books are summarized, and their relation to the material in this book is established. Examples are: completely monotonic functions and general Tauberian theorems. This procedure is commendable in that it gives the reader a comprehensive view of the subject without extending the treatise beyond reasonable bounds. The exposition is very detailed and clear, and many examples and computations are given. In consequence, the book is unusually easy to read and affords a good opportunity for the reader to learn the mathematical techniques employed, if he is not already familiar with them. Great care has been taken to prove each theorem in all the forms which are required for applications. In consequence, the book should have great value as a reference. A brief summary of the contents follows.

Chapter 1. The fundamental concepts of function spaces and functional transformations. Chapter 2. The abscissa of convergence and absolute convergence, the bilateral Laplace transform, uniqueness theorems, composition theorems. Chapter 3. The abscissa of uniform convergence, the analyticity of the generating function, the behaviour of the generating function of the abscissa of convergence and at infinity. Chapter 4. The complex inversion formula, Fourier's theorem, the deformation of the path of integration. Chapter 5. A formula for the partial integral of the Laplace transform. Chapter 6. Parseval's equation. Chapter 7. Conditions for the representation of a function as a Laplace transform. Chapter 8. Additional inversion formulas for the Laplace transform. Chapter 9. The Cesàro means of the Laplace integral. Chapter 10. The Laplace transform of entire functions of exponential type, Borel summability. Chapter 11. The bilateral Laplace transform and the Mellin transform applied to analytic functions. Chapter 12. The Laplace transform of functions belonging to $L_2(0, \infty)$. Chapter 13. Abelian theorems on the behaviour of the Laplace transform in the neighborhood of a finite singular point. Chapter 14. Abelian theorems on the behaviour of the Laplace transform at infinity. Chapter 15. Abelian theorems for the complex inversion formula. Chapter 16. Tauberian theorems. There is an appendix containing detailed statements of various theorems assumed in the text. The historical and bibliographical references are collected in a section following the appendix. The volume concludes with a list of books and papers. *I. I. Hirschman, Jr.*

Lions, Jacques-Louis. Supports de produits de composition. I. C. R. Acad. Sci. Paris **232**, 1530–1532 (1951).

Si F est un ensemble fermé de l'espace euclidien R^n, on appellera $c(F)$ l'enveloppe convexe fermée de F. Théorème: Si S et T sont deux fonctions ou distributions sur R^n, de supports compacts A et B, et si C est le support du produit de composition $S*T$, on a $c(C)=c(A)+c(B)$ (c.à.d. $c(C)$ est la réunion des points $x+y$, $x \in c(A)$, $y \in c(B)$), ce qui généralise un résultat connu pour la dimension $n=1$ [voir par ex. Dufresnoy, mêmes C. R. **225**, 857–859 (1947); ces Rev. **9**, 237]. Corollaire: L'algèbre (pour le produit de composition) des distributions à support limité à gauche par rapport à un cône convexe Γ de R^n [Schwartz, Théorie des distributions, tome II, Actualités Sci. Ind., no. 1122, Hermann, Paris, 1951, p. 32; ces Rev. **12**, 833] est sans diviseur de zéro. La démonstration utilise la transformation de Fourier. Elle donne accessoirement le corollaire suivant: soit $F(z_1, z_2, \cdots, z_n)$ une fonction analytique entière de type exponentiel, à croissance polynomiale sur R^n; soit $n_1(r; z_2, z_3, \cdots, z_n)$ le nombre de ses zéros en z_1, dans le cercle $|z_1| \leqq r$, pour $\zeta=(z_2, z_3, \cdots, z_n)$ fixé; alors la limite $\lim_{r\to\infty} r^{-1}n_1(r; \zeta)$ existe pour tous les ζ, et, exception faite d'un ensemble au plus dénombrable de points ζ, elle est indépendante de ζ. L'article contient de nombreuses erreurs typographiques. 3ème ligne avant la prop. 2, remplacer "$z_1 \in I$" par "de $z_1 \in K$". La prop. 2 doit être remplacée par ceci: "Soit $n(r, z_2)$ le nombre de zéros de $F(z_1, z_2)$ considérée comme fonction en z_1, situés dans $|z_1| \leqq r$. On a $\lim_{r\to\infty} r^{-1}n(r, z_2)=\pi^{-1}(b-a)$, pour tout z_2 non-$\in D$ (D, ensemble dénombrable du plan H_{z_2}) . . .". *L. Schwartz* (Nancy).

Lions, Jacques-Louis. Supports de produits de composition. II. C. R. Acad. Sci. Paris **232**, 1622–1624 (1951).

Cette note généralise la note analysée ci-dessus. Les distributions T telles que, quelle que soit la constante $k>0$, la distribution $T \exp(k\sum_{i=1}^{n} x_i^2)$ soit bornée sur R^n, forment une algèbre $\mathfrak{A}$ pour le produit de composition. Si S et T sont dans $\mathfrak{A}$, de supports (non nécessairement compacts) A et B, si C est le support de $S*T$, on a $c(C)=\overline{c(A)+c(B)}$ (adhérence de $c(A)+c(B)$). *L. Schwartz* (Nancy).

Jarník, V. Sur le produit de composition de deux fonctions continues. Studia Math. **12**, 58–64 (1951).

L'auteur donne un exemple de deux fonctions continues $x(t)$, $y(t)$, sur $(0, 1)$, pour lesquelles la fonction $z(t)=\int_0^t x(t-\tau)y(\tau)d\tau$ est partout sans dérivée pour $t>0$, et même:

$$\limsup_{h\to 0} \frac{z(t+h)-z(t)}{h}=+\infty, \quad \liminf_{h\to 0} \frac{z(t+h)-z(t)}{h}=-\infty$$

(h de signe quelconque) pour tout $t>0$. Cette même propriété est vérifiée pour tous les couples x, y, de deux fonctions continues, sauf ceux d'un ensemble de première catégorie de $C\times C$ (C=espace de Banach des fonctions continues sur $(0, 1)$). *L. Schwartz* (Nancy).

Mikusiński, J. G.-, et Ryll-Nardzewski, Cz. Sur le produit de composition. Studia Math. **12**, 51–57 (1951).

Série de remarques simples sur les propriétés de continuité et différentiabilité du produit de composition de deux fonctions, à partir des propriétés de chacune d'elles. *L. Schwartz* (Nancy).

Tricomi, F. G. On the finite Hilbert transformation. Quart. J. Math., Oxford Ser. (2) **2**, 199–211 (1951).

The author's purpose is to give precise conditions for the resolution of the airfoil equation

$$(*) \qquad f(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)}{y-x}dy=\mathfrak{f}_*[\phi(y)] \quad (-1<x<1).$$

It is shown that if $\phi_1(x)$ and $\phi_2(x)$ belong in $(-1, 1)$ to L^{p_1} and L^{p_2}, respectively ($p_1>1$, $p_2>1$), and if $p_1^{-1}+p_2^{-1}<1$, then

$$\mathfrak{f}\{\phi_1\mathfrak{f}[\phi_2]+\phi_2\mathfrak{f}[\phi_1]\}=\mathfrak{f}[\phi_1]\mathfrak{f}[\phi_2]-\phi_1\phi_2,$$

a result which is also valid for the Hilbert transform. Applying this result the author shows that if $(*)$ has a solution of

class L^p $(p>1)$ then the solution is necessarily of the form

$$(**) \quad \phi(x) = -(1-x^2)^{-\frac{1}{2}} f_a[(1-y^2)^{\frac{1}{2}} f(y)] + C(1-x^2)^{-\frac{1}{2}}.$$

In addition, if $f(x)$ belongs to the class L^p with $p>\frac{4}{3}$, then $\phi(x)$ defined by (**) is a solution of the airfoil equation (*).

Various formulas are established for

$$f_a[(1-t)^\alpha(1+t)^\beta p_n^{(\alpha,\beta)}(t)],$$

where $P_n^{(\alpha,\beta)}$ is a Jacobi polynomial. A particular case leads to the result

$$f_a\left[\left(\frac{1-y}{1+y}\right)^\alpha\right] = \cot \alpha\pi \left(\frac{1-x}{1+x}\right)^\alpha - \frac{1}{\sin \alpha\pi} \quad (0<|\alpha|<1).$$

This formula suggests that if $\phi(x)$ becomes infinite like $A(1-x)^{-\alpha}$ or $A(1+x)^\alpha$ as $x \to \pm 1$, then $f(x)$ behaves similarly. A theorem of this sort is established.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Polynomials, Polynomial Approximations

Ostrowski, A. Sur une règle de Laguerre. Ann. Mat. Pura Appl. (4) **31**, 65–68 (1950).

For a real polynomial

$$f(x) = f_n(x) = a_0x^n + \cdots + a_n, \quad a_0 \neq 0,$$

an upper bound to the number of zeros in an interval $x>\alpha>0$ may be found by either the Laguerre rule or the Budan-Fourier rule. The former rule consists in introducing the sequence

$$f_k(x) = a_0x^k + \cdots + a_k, \quad k=0, 1, \cdots, n-1;$$

the number of zeros being then at most

$$L(\alpha) = V(f_n(\alpha), f_{n-1}(\alpha), \cdots, f_0(\alpha))$$

where $V(u_1, u_2, \cdots, u_k)$ denotes the number of variations of sign in the non-zero terms of the sequence $u_1, u_2, \cdots, u_k$. The latter rule employs the derivatives $f^{(k)}(x)$ of $f(x)$; the number of zeros being at most

$$B(\alpha) = V(f(\alpha), f'(\alpha), \cdots, f^{(n)}(\alpha)).$$

In the present note, Ostrowski gives for Laguerre's theorem a new proof which incidentally shows that $D(\alpha) = L(\alpha) - B(\alpha)$ is always a non-negative even number when $f(\alpha) \neq 0$ or when both $f(\alpha)=0$ and an odd number of zeros of $f(x)$ lie between $+0$ and $\alpha+0$, but that $D(\alpha)$ is a positive odd number when both $f(\alpha)=0$ and an even number of zeros lie between $+0$ and $\alpha+0$. *M. Marden* (Milwaukee, Wis.).

Kuipers, L. A property of the graph of some special real functions. Nieuw Arch. Wiskunde (2) **23**, 243–246 (1951).

The author proves the following theorems: (1) Let $f(x)$ be the function $(x-a)(x-b)\phi(x)$ $(a<b)$, where $\phi(x)$ is a differentiable function on (a, b) which satisfies $\phi(x)<0$ on $a \leqq x \leqq b$, and $\phi(a)\phi(b) \leqq [\phi(\xi)]^2$ $(\xi = \frac{1}{2}(a+b))$. Let the tangents at a and b meet at a point T with y_T as ordinate, and let $f(x)$ reach its maximum value on $a \leqq x \leqq b$ at $x=m$. Then $y_T \leqq 2f(m)$, where equality holds only if $m=\xi$, $\phi(a)=\phi(b)$ and $\phi(a)\phi(b) = [\phi(\xi)]^2$. (2) Let $P(x)$ be a polynomial with real coefficients. Let $P(x)$ have $2k$ imaginary zeros and $n-2k$ real zeros $(k \geqq 1, n \geqq 2k+2)$. Let a_{p-1} and a_p be consecutive real zeros $(a_{p-1}<a_p)$. Let the tangents at a_{p-1} and a_p meet at a point T_p with y_{T_p} as ordinate, and let $|P(x)|$ reach its maximum value on $a_{p-1} \leqq x \leqq a_p$ at $x=m_p$. Furthermore, if z_1 and $\bar{z}_1$ (the conjugate of z_1) are the complex zeros, let these numbers (located in the complex plane) be points of G, where G is the region which the interiors of the $n-2k-1$ hyperbolas $(x-\xi_p)^2 - y^2 = \frac{1}{4}(a_p - a_{p-1})^2 a_{p-1}$ $(\xi_p = \frac{1}{2}(a_{p-1}+a_p))$ have in common. Then $|y_{T_p}| < 2P(m_p)$, $p=2, 3, \cdots, n-2k$. These theorems extend the theorem due to G. Szekeres [cf. Erdös and Grünwald, Ann. of Math. (2) **40**, 537–540 (1939); these Rev. **1**, 1] concerning polynomials with only real roots. *E. Frank* (Chicago, Ill.).

Householder, Alston S. Polynomial iterations to roots of algebraic equations. Proc. Amer. Math. Soc. **2**, 718–719 (1951).

If ξ is a root of the equation $f(x)=0$ and if $\phi(x)$ satisfies $\phi(\xi)=\xi$, $\phi^{(s)}(\xi)=0$, $s=1, 2, \cdots, r-1$, then $\phi(x)$ is said to define an iteration of order r to the root ξ. The following algorithm is described: Let $f(x)$ be a polynomial with no multiple factors; let $p(x)$ and $q(x)$ be any polynomials satisfying $pf'+qf \equiv 1$; let $\phi_1 = x - p_1 f$, $p_1 = p$, $\phi_r = \phi_{r-1} + p_r(-f)^r/r!$ $p_r = pp'_{r-1} - (r-1)qp_{r-1}$. Then $\phi_r(x)$ is clearly a polynomial, and it defines an iteration of order $r+1$ to any root ξ of $f(x)=0$. *E. Frank* (Chicago, Ill.).

Hahn, Wolfgang. Über die Zerlegung einer Klasse von Polynomen in irreduzible Faktoren. Math. Nachr. **3**, 327–329 (1950).

In the adjustment of observations by least squares N. E. Nörlund considered the polynomials

$$\varphi_{n,r}(x) = \prod_{\nu=1}^{n-1}\left(x - \frac{\sin r\nu\pi/n}{\sin \nu\pi/n}\right).$$

W. Jenne showed that it is desirable to know the factorization of these polynomials into irreducible factors in the domain of rational numbers. The present author gives the exact factorization and verifies Jenne's conjectures regarding the factors. *W. E. Milne* (Corvallis, Ore.).

Merli, Luigi. Su una classe di polinomi interpolanti costruiti con punti fondamentali normalmente distribuiti. Boll. Un. Mat. Ital. (3) **6**, 103–106 (1951).

Soient $\{x_k^{(n)}\}$ $(k=1, 2, \cdots, n;\ n=1, 2, \cdots)$ une suite de points normalement distribués, sur $-1 \leqq x \leqq 1$ au sens de Féjer [Math. Ann. **106**, 1–55 (1932)], $l_k^{(n)}(x)$ le polynome de degré $\leqq n-1$ qui prend en $x_k^{(n)}$ la valeur 1 et s'annule pour $x_i^{(n)}$ $(i \neq k)$; alors, pour toute fonction continue $f(x)$ on a:

$$\lim_{n\to\infty} \sum_{k=1}^{n} [1-(x_k^{(n)})^2] f(x_k^{(n)}) [l_k^{(n)}(x)]^2 = (1-x^2)f(x)$$

uniformément pour $-1 \leqq x \leqq 1$. *J. Favard* (Paris).

Szász, Otto. Identities and inequalities concerning orthogonal polynomials and Bessel functions. J. Analyse Math. **1**, 116–134 (1951).

The inequality

$$\Delta_n(x) = (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) \geqq 0, \quad -1 \leqq x \leqq 1, \quad n=1, 2, 3, \cdots,$$

due to P. Turan in the case of Legendre polynomials has been extended to ultraspherical, Laguerre, and Hermite polynomials by Szegö [Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429]. Corresponding inequalities are proved here by an elementary method based on the recurrence formula for the ultraspherical polynomials, the generalized Laguerre polynomials, and for Bessel functions. The author also derives some estimates of the expression

Δ_n in these cases based on positive representations of this expression. *M. J. Gottlieb* (Chicago, Ill.).

Hahn, Wolfgang. Über lineare Differentialgleichungen, deren Lösungen einer Rekursionsformel genügen. Math. Nachr. **4**, 1–11 (1951).

A polynomial chain is a sequence of polynomials ϕ_n which satisfy a three-term recursion formula

$$\phi_n(x)=(x-\alpha_n)\phi_{n-1}(x)-\beta_n\phi_{n-2}(x), \quad \beta_n\neq 0.$$

A sequence of orthogonal polynomials is the special case where α_n is real and β_n positive. Under certain general assumptions it is shown that if each polynomial ϕ_n in a chain satisfies a linear homogeneous differential equation of order r (independent of n) with polynomials as coefficients, then (1) the singularities of these differential equations do not depend on n, and (2) the characteristic equation corresponding to each singularity does not depend on n. These results are used to give simple proofs of some results of Krall [Duke Math. J. **4**, 705–718 (1938)] concerning polynomial chains satisfying a particular sequence of differential equations. Finally two methods are given by which new chains can be constructed from given ones. *E. A. Coddington.*

Special Functions

Vinti, John P. Note on a series of products of three Legendre polynomials. Proc. Amer. Math. Soc. **2**, 19–23 (1951).

The author evaluates the sum of an infinite number of products of Legendre functions which are all of the same index, but of different variables. The result is a function of the three variables in question. A geometric interpretation is given for this function. *M. J. O. Strutt* (Zurich).

Karamata, J., und Tomić, M. Über die asymptotische Formel für die Legendresche Polynome. Srpska Akad. Nauka. Zbornik Radova, Knj. **7**. Matematički Institut, Knj. **1**, 64–72 (1951). (Serbo-Croatian. German summary)

The asymptotic formula for Legendre polynomials of large degree is usually proved either from the generating function (by Darboux's method) or else from an integral representation. In this paper the authors give a new proof which is based on the Fourier sine expansion of $P_n(\cos\theta)$. *A. Erdélyi* (Pasadena, Calif.).

Forster, Herbert. Über das asymptotische Verhalten der Besselschen Funktion längs einer Welle der durch $z=J_\lambda(x)$ im $x-\lambda-z$-Raum definierten Wellenfläche. Math. Z. **54**, 217–253 (1951).

For positive x and y, $z=J_y(x)$ is the equation of a corrugated surface. Let $x_1(y)$ be the first positive zero of $J_y(x)=0$, and let y_0 be any fixed positive number. The author investigates $J_y(x)$ for $y\geqq y_0$, $x\geqq x_1(y)$, and in particular the behaviour of $J_y(x)$ as $x, y\to\infty$ "along" a wave of the surface $z=J_y(x)$. Since $x_1>y$, one may put $y/x=\cos\alpha$. Theorem 1 states that

$$\tan^{\frac{1}{2}}\alpha\cdot y^{\frac{1}{2}}J_y(x)=K_1(y,x)\cos[y(\tan\alpha-\alpha)]+K_2(y,x)\sin[y(\tan\alpha-\alpha)],$$

where K_1 and K_2 are positive functions, bounded, and bounded away from zero. Theorem 2 states inequalities satisfied by $y(\tan\alpha-\alpha)$ on the nth crest and in the nth trough of the wave surface, and also for the nth zero of $J_y(x)$. Theorem 3 gives bounds for $x^{1/2}n^{1/6}J_y(x)$ in the region of the nth wave. These results are obtained from Sommerfeld's integral representation of Bessel functions by a technique similar to the method of steepest descents except that asymptotic expansions are replaced by appraisals and inequalities. [Reviewer's remark: The author seems to be unacquainted with the more recent work on asymptotic expansions with remainders (C. S. Meijer) and on uniform asymptotic expansions (R. E. Langer, T. M. Cherry).] *A. Erdélyi* (Pasadena Calif.).

Lambe, C. G. Polynomial expressions for Lamé functions. Quart. J. Math., Oxford Ser. (2) **2**, 53–59 (1951).

When Lamé functions are squared they appear as polynomials which are solutions of a differential equation of the third order. The author shows that squared Lamé functions and also Lamé products can be expressed in terms of the polynomial solutions of a second order differential equation. He starts from the algebraic form of the Lamé differential equation and states the third order equation mentioned above. Differentiating the complete expression for the solution of this equation $(n+1)$ times, he obtains a second order differential equation for the nth differential coefficient of this expression. The equation obtained is of Heun's form. Another second order equation is obtained similarly by differentiating the third order equation n times. Halphen's transformation is applied to obtain from the original Lamé equation a second order equation similar to the one just mentioned. Recurrence relations and integral equations are then considered. *M. J. O. Strutt* (Zurich).

Ogilvy, C. S. The Beta function. Amer. Math. Monthly **58**, 475–479 (1951).

Auner, Michael. Eine Anwendung des Verfahrens der sukzessiven Approximationen auf Differenzengleichungen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 124–125 (1950).

The author shows how the terms in the asymptotic expansion of the gamma function, (or of the incomplete gamma function), can be computed by iteration using the difference equation satisfied by the gamma function (or incomplete gamma function). *W. E. Milne* (Corvallis, Ore.).

Fulks, W. A generalization of Laplace's method. Proc. Amer. Math. Soc. **2**, 613–622 (1951).

Der Verf. untersucht unter bestimmten Voraussetzungen der Funktionen $f(t)$, $\Phi(t)$ und $\psi(t)$ das asymptotische Verhalten des Integrals

$$g_{h,k}=\int_a^b f(t)e^{-h\Phi(t)+k\psi(t)}dt$$

wenn $h, k\to+\infty$, $k=o(h)$. Es handelt sich um die Übertragung der Laplace'schen Methode für Integrale mit einem Parameter. Der Verf. gibt Anwendungen seiner Resultate auf die unvollständige Γ-Funktion, $\Gamma(a+1,x)=\int_x^\infty e^{-u}u^a du$. Mit Hilfe der Substitutionen $u=x(t+1)$, $a=x+y$, $y=o(x)$ wird dieses Integral auf die obige Form transformiert, sodass die erhaltene asymptotische Darstellung direkt angewendet werden kann. Der Verf. beweist, dass

$$\Gamma(x+y+1)\sim(\tfrac{1}{2}\pi)^{\frac{1}{2}}e^{-x}x^{x+y+\frac{1}{2}}.$$

W. Saxer (Zürich).

Ritchie, R. H. On a definite integral. Math. Tables and Other Aids to Computation 4, 75–77 (1950).

Using Laplace transforms the author proves that the function

$$f(x)=\int_0^\infty (u+x)^{-1}\exp(-u^2)du,$$

tabulated by Goodwin and Staton [Quart. J. Mech. Appl. Math. 1, 319–326 (1948); these Rev. 10, 268] can be written as $f(x)=\frac{1}{2}\exp(-x^2)\,\mathrm{Ei}(x^2)+\pi^{\frac{1}{2}}F(x)$, where $\mathrm{Ei}(y)=\int_{-\infty}^{y}u^{-1}e^u du$ and $F(y)=\exp(-y^2)\int_0^y \exp(u^2)du$ are well-tabulated functions. *J. G. van der Corput.*

Robinson, John E. Note on the Bose-Einstein integral functions. Physical Rev. (2) 83, 678–679 (1951).

The so-called Bose-Einstein function

$$\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^{x+\alpha}-1}dx$$

is well known to mathematicians as a special Lerch zeta function. [See, for instance, Truesdell, Ann. of Math. (2) 46, 144–157 (1945); these Rev. 6, 152, and the references given there.] In this note the author derives Lindelöf's expansion in powers of α [equation (13) in Truesdell's paper]. *A. Erdélyi* (Pasadena, Calif.).

Breit, G., and Hull, M. H., Jr. Asymptotic expansion of irregular Coulomb function for angular momentum zero. Physical Rev. (2) 80, 392–395 (1950).

Yost, Wheeler and Breit [same Rev. (2) 49, 174–189 (1936)] have proved that in the limit of zero energy the irregular Coulomb function G_L, where $L\hbar$ denotes the angular momentum, possesses the property that $\theta_L=G_L/(D_L\rho^{-L})$ approaches a finite limit which is, apart from an elementary factor, equal to $K_{2L+1}(x)$; here $K_\nu(x)$ is the Bessel function of an imaginary argument of the second kind. They stated that the replacement of K_{2L+1} by a series obtainable from the series for the regular Coulomb function by the substitution of the K_ν for the Bessel functions of an imaginary argument I_ν leads to values which agree in some cases with results obtained by direct calculation also at finite values of the energy. In the special case that the angular momentum $L\hbar$ is equal to zero, Breit and Bouricius [ibid. (2) 75, 1029–1050 (1949)] have verified this expansion for a number of terms. This paper contains a justification for that expansion in the case $L=0$, and a general rule of replacement of the Bessel functions of the first kind with an imaginary argument in the corresponding expansion of the regular Coulomb function is established. Finally the authors show that the formulas developed are convenient for the calculation of coefficients of the energy. *J. G. van der Corput.*

Bloch, I., Hull, M. H., Jr., Broyles, A. A., Bouricius, W. G., Freeman, B. E., and Breit, G. Coulomb functions for reactions of protons and alpha-particles with the lighter nuclei. Rev. Modern Physics 23, 147–182 (1951).

Coulomb functions are solutions of the differential equation

$$\frac{d^2y}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right]y=0.$$

The regular Coulomb function F_L is that solution which behaves asymptotically as $\sin(\rho-\frac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L)$, the irregular Coulomb function G_L that solution which behave as $\cos(\rho-\frac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L)$, when $\rho\to\infty$. Here $\sigma_L=\arg\Gamma(L+1+i\eta)$. Amplitude and phase are defined by $F_L=A_L\sin\varphi_L$, $G_L=A_L\cos\varphi_L$, with the additional requirement that $\varphi_L=0$ when $\rho=0$. The paper contains numerical tables for $L=0(1)4$, designed to provide values with an accuracy of one percent. The 39 tables give values of F_L, A_L, φ_L, and of certain auxiliary functions with convenient interpolation properties. The independent variables are $\log_{10}\eta=-.8(.1).6$ and ρ, $0<\rho\leqq 6.2$. Each table contains a note on accuracy and interpolation. *A. Erdélyi.*

Källén, Gunnar. The second approximation of the asymptotic phase for the Yukawa potential, treated with Laplace-transformations. Ark. Fys. 2, 33–46 (1950).

If the interaction potential between two particles is denoted by $\lambda\varphi(r)$, where λ is the coupling constant, the radial part $R=fr^{l}$ of the wave function satisfies the equation $r(f''+k^2f)+2(l+1)f'+\lambda\varphi rf=0$. It is sufficient to consider only real solutions, as both the real and the imaginary parts of a complex solution satisfy the equation. Substituting $f=\sum_0^\infty\lambda^n f_n$ we obtain the recurrence relation

$$r(f_n''+k^2f_n)+2(l+1)f_n'+r\varphi f_{n-1}=0.$$

If $r\varphi f_{n-1}=\int_C e^{pr}h_{n-1}(p)dp$, where C is some integration path, we can write the solutions $f_n(r)$ of the recurrent relation as integrals along C; for $n\geqq 1$ the integrand contains the factor $h_{n-1}(p)$. The fact that we restrict ourselves to real solutions furnishes the contour C and the fact that $f_0(r)$ is finite at the origin gives

$$f_0(r)=A_0\int e^{pr}(p^2+k^2)\log(1+p^2/k^2)dp,$$

taken along a lemniscate around the point $-ik$ in positive and around the point ik in negative sense. This gives the asymptotic behavior of $f_0(r)$ for large positive r and the constant is determined by the condition that the amplitude tends to 1 as $r\to\infty$.

To treat $f_n(r)$ for $n\geqq 2$ it is necessary to calculate $h_{n-1}(p)$, which the author does in the special case $r\varphi(r)=e^{-pr}$. In that case the author obtains the first (Born's) approximation, namely, the behavior of $f_1(r)$ for large r and also the second approximation of $f_2(r)$. In this paper the explicit calculations are carried out only to the second approximation and for small angular moments. The numerical results are compared with earlier approximation formulae of Pais and Ramsey and with the more accurate calculations made by means of the variational method of Hulthén.

J. G. van der Corput (Stanford University, Calif.).

Harmonic Functions, Potential Theory

Tôki, Yukinari. A topological characterization of pseudo-harmonic functions. Osaka Math. J. 3, 101–122 (1951).

Pseudo-harmonic functions u introduced by the reviewer and M. Heins [see M. Morse, Topological methods in the theory of functions of a complex variable, Princeton Univ. Press, 1947; these Rev. 9, 20] are topologically characterized by the author on a separable 2-manifold F. The sets L_c at the respective u-levels c are by definition equi-locally connected at a point $p\in F$ when for any neighborhood N_p of $p\in F$ there exists a $N_p'\subset N_p$ such that any pair of points in $L_c\cap N_p'$ are in a connected subset of $L_c\cap N_p$. Then (Th. 1) u is pseudo-harmonic on F if and only if u is continuous and open on F, with the sets L_c equi-locally connected on F, possibly excepting a discontinuum in F. By definition v, defined over F, is pseudo-conjugate to u if for some neigh bor-

hood N_p of each $p \in F$, there exists a homeomorphic mapping T of $\{|z|<1\}$ onto N_p such that $v(T(z))$ is the harmonic conjugate of $u(T(z))$. Th. 2 affirms that v is pseudo-conjugate to a given pseudo-harmonic u if and only if v is continuous and open, and if on no continuum in a u-level set $v(p)$ is a constant. The reviewer verifies the necessity of the conditions of Th. 2 but the example $u=y$, $v=y-x^2$ shows that the conditions are insufficient. Th. 3, affirming the existence of a v, pseudo-conjugate to u, appears unproved, since the curves $C_1, C_2, \cdots$ described on page 118 are not always (as affirmed) level curves of a pseudo-harmonic function. An example will show that $C_1, C_2, \cdots$ need not be equi-locally connected. Th. 2 is used in the proof of Th. 3 and of later theorems. *M. Morse.*

Višik, M. I. On an inequality for the boundary values of harmonic functions in a sphere. Uspehi Matem. Nauk (N.S.) 6, no. 2(42), 165–166 (1951). (Russian)

Let u be harmonic in a three-dimensional sphere with radius R and continuously differentiable in the closed sphere, let r, θ, φ be spherical coordinates and put $u_r=\partial u/\partial r$, etc. Proof of the inequality

$$\int_S u_R^2 dS \leqq R^{-2}\int_S (u_\theta^2+\sin^{-2}\theta u_\varphi^2)dS$$

where S is the surface of the sphere and dS is the surface element. Indications of generalizations. *L. Gårding.*

*Diaz, J. B. **Upper and lower bounds for quadratic functionals.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 279–289. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Vorgelegt sei die Aufgabe $\Delta v=f(x, y)$, wobei v in einem Bereich D regulär sein und längs des Randes C von D verschwinden soll. Ziel der Arbeit ist, für $\int\int_D (v_x^2+v_y^2)dxdy$ obere und untere Schranken zu gewinnen, die durch ein fortschreitendes Verfahren ständig verbessert werden können. Es wird

$$\int\int_D (\varphi_x\psi_x+\varphi_y\psi_y)dxdy=(\varphi, \psi)$$

als skalares Produkt der "Vektoren" φ und ψ aufgefasst und es werden Ungleichungen vom Typus der Schwarzschen und Besselschen Ungleichung gewonnen. Sinngemässe Anwendung der Besselschen Ungleichung auf das vorliegende Problem liefert unter Zuhilfenahme der Greenschen Formel

$$\sum_{i=1}^{p}\left(\int\int_D w_i\Delta v dxdy\right)^2 \leqq \int\int_D (v_x^2+v_y^2)dxdy$$

$$\leqq \int\int_D (z_x^2+z_y^2)dxdy-\sum_{i=1}^{q}\left(\int\int_D [z_x z_{ix}+z_y z_{iy}]dxdy\right)^2.$$

Dabei bedeuten w_i und z_i Orthogonalsysteme (im Sinne der oben eingeführten Massbestimmung). w_i genügt der Bedingung $w_i=0$ auf C, die z_i genügen der Gleichung $\Delta z_i=0$ und z genügt der oben angeführten partiellen Differentialgleichung für v. Für $f(x, y)=-2$ ergeben sich unter Zuhilfenahme der Schwarzschen Ungleichung die bekannten Minimalprinzipien für die Lösung der Saint-Venant'schen Torsionsproblems. *P. Funk* (Wien).

Synge, J. L. Pointwise bounds for the solutions of certain boundary-value problems. Proc. Roy. Soc. London. Ser. A. 208, 170–175 (1951).

Pointwise bounds for the solutions of boundary value problems of the Dirichlet-Neumann type are obtained by a method which is a compromise between the method of Diaz and Greenberg [J. Math. Physics 27, 193–201 (1948); Quart. Appl. Math. 6, 326–331 (1948); these Rev. 10, 213, 167] and Maple [ibid. 8, 213–228 (1950); these Rev. 12, 704] and Synge [Proc. Roy. Soc. London. Ser. A. 191, 447–467 (1947); Proc. Roy. Irish Acad. Sect. A. 53, 41–64 (1950); these Rev. 10, 81; 12, 556]. In this modified procedure the solutions are located on a hypercircle in function space using the method previously developed by Synge, and the pointwise bounds are obtained by taking the scalar product of the solution with certain Green's vectors. From this point on, the present paper follows essentially the Diaz-Greenberg method: the divergence of the scalar products is avoided by introducing additional functions which match the Green's functions on the boundary but are regular inside. This replaces the Maple-Synge technique of cutting out a spherical region centered at the point where bounds are sought. The method is illustrated by applying it to a Dirichlet problem to obtain bounds for its solution and for the first and second derivatives of the solution.

C. G. Maple (Washington, D. C.).

Rapoport, I. M. On a problem of potential theory. Ukrain. Mat. Žurnal 2, no. 2, 48–55 (1950). (Russian)

The author gives an elementary solution for determining that (star-shaped) contour L, which possesses a prescribed exterior logarithmic potential

$$V(r, \varphi)=\alpha_0 \log r+\sum_{m=1}^{n}(\alpha_m \cos m\varphi+\beta_m \sin m\varphi)r^{-m}.$$

He writes the equation of boundary in the form

$$z=f[\exp(i\vartheta)],$$

and sets

$$f(\zeta)=\int_0^\zeta P_n^2(\zeta)d\zeta,$$

where $P_n(\zeta)=\sum_{\nu=0}^{n}a_\nu\zeta^\nu$. The solution of the problem is then reduced to the solution of a finite system of algebraic equations in the a_ν, $\nu=0, 1, \cdots, n$. *S. Bergman.*

Savin, S. A. On an integral of the two-dimensional Laplace equation. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 392 (1951). (Russian)

The author seeks solutions of the Laplace equation

$$\nabla^2 V(x, y)\equiv \partial^2 V/\partial x^2+\partial^2 V/\partial y^2$$

of the form $V(x, y)=F(u)=F(\alpha x+\epsilon y+\rho r)$, where $r^2=x^2+y^2$ and α, ϵ, ρ are constants. Computation and the specialization $\alpha^2+\epsilon^2=\rho^2$ give $\nabla^2 V=\rho r^{-1}(2uF''+F')$, so that for $\rho^2=\alpha^2+\epsilon^2=0$ we have the solution $V(x, y)=F(x\pm iy)$, while otherwise $F''/F'+1/2u=0$ leads to the solution $V(x, y)=C(x\cos\theta+y\sin\theta+r)^{\frac{1}{2}}$, where C and θ are arbitrary constants. *E. F. Beckenbach* (Princeton, N. J.).

Trjitzinsky, W. J. Mixed Laplacians and potential representations. Ann. Mat. Pura Appl. (4) 31, 143–230 (1950).

Étude détaillée et comparée de laplaciens (du genre de celui de Zaremba) généralisant la dérivée seconde directe. Conditions correspondantes de validité de la formule de Poisson et d'intégration de $\Delta u=f$, c'est-à-dire de représenta-

tion potentielle de u au moyen de la densité Δu. Ce genre d'étude, pas plus que les anciennes recherches de Petrini sur le potentiel, ne parait plus aujourd'hui bien attrayante et utile, maintenant que les distributions et le laplacien de Schwartz donnent souplesse et généralité aux théories classiques. Signalons toutefois des développements sur les conditions d'harmonicité analogues aux conditions d'holomorphie de Looman-Menchoff. *M. Brelot* (Grenoble).

Poritsky, H. Extension of Weyl's integral for harmonic spherical waves to arbitrary wave shapes. Comm. Pure Appl. Math. **4**, 33–42 (1951).

According to Weyl [Ann. Physik (4) **60**(365), 481–500 (1919)], a sinusoidal spherical wave can be represented in terms of sinusoidal plane waves, viz.

$$\frac{e^{-ikR}}{R}=-\frac{ik}{2\pi}\int \exp\left[-ik(\alpha x+\beta y+\gamma z)\right]d\omega,$$

where the integral is to be taken over a certain part of the complex unit sphere. If both sides are multiplied by the time factor $\exp(ikct)$, Weyl's identity can be rewritten as

$$(*)\qquad \frac{F(ct-R)}{R}=-\frac{1}{2\pi}\int F'\left[ct-(\alpha x+\beta y+\gamma z)\right]d\omega$$

where $(**)$ $F[z]=\exp[ikz]$ and $F'=(d/dz)F(z)$. The author is concerned with the question whether $(*)$ holds for functions other than $(**)$. Two heuristic proofs of $(*)$ are presented which, according to the author, will be found insufficient on closer examination. The main difficulty is that $(*)$ requires the analytic continuation of a function F originally known for real values of the argument. Let $F(\xi)$ denote a real function defined for $-\infty<\xi<\infty$. Then

$$\varphi(\xi,\eta)=\frac{\eta}{\pi}\int_{-\infty}^{\infty}\frac{F(\xi')}{(\xi'-\xi)^2+\eta^2}d\xi' \qquad (\eta>0)$$

is a two-dimensional potential function such that

$$F(\xi)=\varphi(\xi,+0).$$

Let $\psi(\xi,\eta)$ denote the conjugate potential function, and let $F_1(z)=\varphi(\xi,\eta)+i\psi(\xi,\eta)$, $z=\xi+i\eta$. Then $F_1(z)$ is an analytic function of z when $\eta\geqq 0$, while $F(\xi)=\Re[F_1(\xi+i0)]$. The author then finds, from $(*)$ applied to F_1, that

$$\frac{F(ct-R)}{R}=\Re\left\{-\frac{1}{2\pi}\int F_1'\left[ct-(\alpha x+\beta y+\gamma z)\right]d\omega\right\}.$$

Application of this identity to Heaviside's unit function appears possible after a slight change of Weyl's domain of integration. [The reviewer observes that under certain conditions one has simply

$$F_1(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{F(\xi')}{\xi'-z}d\xi',\quad F_1'(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{F(\xi')}{(\xi'-z)^2}d\xi'.$$

The x and y axes in Fig. 1 should be interchanged.]
C. J. Bouwkamp (Eindhoven).

Buehler, Robert J., and Hirschfelder, Joseph O. Bipolar expansion of Coulombic potentials. Physical Rev. (2) **83**, 628–633 (1951).

Let P_1 and P_2 be two points in three-dimensional space, P_1 referred to a system of spherical polar coordinates with pole O_1, and P_2 to a system with pole O_2, it being assumed that O_1O_2 is the polar axis for both systems. In the theory of intermoleculcar forces it is useful to have an expansion of $(P_1P_2)^{-1}$ in terms of spherical harmonics. Such an expansion was obtained by Carlson and Rushbrooke [Proc. Cambridge Philos. Soc. **46**, 626–633 (1950); these Rev. **12**, 259]. The radial coefficients take different forms according to the relative magnitudes of O_1P_1, O_2P_2, and O_1O_2. There are four cases to be distinguished. Three of these have been investigated by Carlson and Rushbrooke. The present author gives all four cases and for the new case (which is the most difficult of the four) the numerical values of coefficients up to and including order 3. *A. Erdélyi.*

Differential Equations

Segers, Jack G. Sur les dérivées d'ordre supérieur des intégrales d'une équation récurro-différentielle. Bull. Soc. Roy. Sci. Liège **20**, 114–119 (1951).

If f_i for $n\leqq i=n+(k-j-1)q+1,\cdots,n+(k-j)q$, $j=1,2,\cdots,k$, is j times continuously differentiable in the set of all its arguments, the component y_n in the solution of the system $y_n'=f_n(x,y_n,y_{n+1},\cdots,y_{n+q})$, $y_n(0)=a_n$ is $k+1$ times continuously differentiable. *J. M. Thomas.*

Szmydtówna, Z. Sur les intégrales premières de l'équation $y'=f(x,y)$. Ann. Soc. Polon. Math. **23**, 167–182 (1950).

L'autore dimostra che per ogni intero n si può costruire un insieme aperto e semplicemente connesse Ω_n e una funzione $f_n(x,y)$ dotata di derivate di ordine comunque elevato, tali che ogni integrale primo dell'equazione $y'=f_n(x,y)$ che sia continuo in Ω_n con tutte le derivate fino a quelle di ordine n incluse sia necessariamente costante in Ω_n.
C. Miranda (Napoli).

Vasil'eva, A. B. On differentiation with respect to a small parameter of solutions of systems of differential equations. Doklady Akad. Nauk SSSR (N.S.) **78**, 845–848 (1951). (Russian)

Vasil'eva, A. B. On the differentiation of solutions of systems of differential equations with respect to the largest of small parameters. Doklady Akad. Nauk SSSR (N.S.) **77**, 781–784 (1951). (Russian)

In the two papers under consideration the author continues and extends an earlier investigation [same Doklady **75**, 483–486 (1950); these Rev. **12**, 412]. Take a system

$$\mu_\alpha\frac{dz_{j_\alpha}}{dt}=F_{j_\alpha}(z_{l_1};\cdots;z_{l_n};y_h;t),\quad \frac{dy_i}{dt}=f_i(z_{l_1};\cdots)$$

with initial values z_j^0, y_i^0 at $t=0$. The μ_α are functions of a parameter μ, positive and $\to 0$ with μ in such manner that $\mu_\alpha/\mu_{\alpha+1}\to 0$ also. In the first paper the derivatives of the solution are taken as to μ_α and shown to tend to certain limiting values in a finite t interval (description extremely complicated) as $\mu\to 0$. The second paper deals with the special case of a system

$$\mu_j\frac{dz_j}{dt}=F_j(z_l;y_h;t),\quad \frac{dy_i}{dt}=f_i(z_l;\cdots)$$

and deals with the same question and derivatives as to μ_r. [Additional relevant reference: A. N. Tihonov, Mat. Sbornik N.S. **27**(69), 147–156 (1950); these Rev. **12**, 181.]
S. Lefschetz (Princeton, N. J.).

Yakubovič, V. A. Criteria of stability for systems of two canonical equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) 78, 221–224 (1951). (Russian)

Let there be given a system

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q} \tag{1}$$

where $2H=\alpha(t)q^2+2\beta(t)pq+\gamma(t)p^2$ and α, β, γ are continuous and periodic with period ω. In an earlier paper [same Doklady 74, 901–903 (1950); these Rev. 12, 413] it was shown that in the space of the forms H, the set of the forms for which all the solutions of (1) are bounded consists of a countable collection O_n ($n=0, \pm1, \pm2, \cdots$) of open connected components, the regions of stability. Let $h_m(t)$, $h_M(t)$ be the smallest and largest characteristic root of $\left\|\begin{smallmatrix}\alpha & \beta\\ \beta & \alpha\end{smallmatrix}\right\|$. We then have: Stability criterium. If

$$n\pi<\int_0^\omega h_m(t)dt\leqq\int_0^\omega h_M(t)dt<(n+1)\pi$$

then $H\varepsilon O_n$.

Let x denote the vector (p, q) and let

$$\varphi_x=\int_0^\omega[2H/(p^2+q^2)]dt.$$

Set also $\tan\theta=q/p$. Oscillation theorem: Let x_1, x_2 be solutions of systems (1) for H_1, H_2 and $x_1(0)=x_2(0)$. If $H_1\leqq H_2$ and inequality holds for at least one t then the vector $x_1(t)$ is in front of the vector $x_2(t)$, $\theta_1(t)\geqq\theta_2(t)$, and $\varphi_{x_1}>\varphi_{x_2}$. Stability criterium 2. If $H_1\leqq H\leqq H_2$ and $H_1, H_2\varepsilon O_n$ then also $H\varepsilon O_n$.

A thorough going oscillation theorem is also given for certain forms H_λ depending upon a parameter λ. There are either no proofs or else only some indications of proofs. [Additional references: M. G. Krein, same Doklady 73, 445–448 (1950); these Rev. 12, 100; N. V. Adamov, Mat. Sbornik (1) 42, 651–668 (1935)]. *S. Lefschetz.*

Saharnikov, N. A. A qualitative picture of the behavior of a trajectory near the boundary of a region of stability containing a singular point in the form of a center. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 349–354 (1951). (Russian)

The following theorem is proved: If A is a center of the system $\dot{x}=P(x, y)$, $\dot{y}=Q(x, y)$ (P, Q entire functions) and if all other singular points are simple, then either (1) all trajectories are closed curves (i.e. trajectories corresponding to periodic solutions) surrounding A; or (2) the boundary of the region swept by all the closed trajectories surrounding A consists of trajectories (a) passing through ∞, or (b) passing through at least one singular point, all the singular points on the boundary being necessarily of the saddle type. *J. L. Massera* (Montevideo).

Butlewski, Zygmunt. Sur les intégrales d'un système d'équations différentielles. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 73–104 (1950). (French. Polish summary)

Let a continuous function $x(t)$ on $0\leqq t<\infty$ be called oscillatory if $x(t)$ changes sign infinitely often for large t and let a pair of continuous functions $(x(t), y(t))$ be called oscillatory if both of the functions are oscillatory. The author establishes necessary conditions and sufficient conditions in order that some or every solution $(x(t), y(t))$ of $x'=f(t, x, y)$, $y'=g(t, x, y)$ be oscillatory. In certain oscillatory cases, he proves the monotony of the sequence of the relative maxima of $|x(t)|$ or $|y(t)|$. Some of the theorems are rather complicated analogues of results on second order linear differential equations and involve conditions too long to state here. Two of the simpler results are as follows: (I) If $f(t, x, 0)\equiv0$, $g(t, 0, y)\equiv0$, $f(t, x, y)\neq0$ for $y\neq0$ and $g(t, x, y)\neq0$ for $x\neq0$, then a solution $(x(t), y(t))$ is oscillatory if and only if one of the components is; in which case, the zeros of $x(t)$ and $y(t)$ are simple and separate each other. (II) If $f(t, x, y)y>0$ for $y\neq0$ and $g(t, x, y)x>0$ for $x\neq0$, then no solution $(x(t), y(t))$ is oscillatory. *P. Hartman.*

Bielecki, Adam. Sur une équation différentielle binome du II-me ordre. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 13–17 (1950). (French. Polish summary)

Let $A(x)$ be a positive, continuous, non-decreasing function on $0\leqq x<\infty$ such that $A(x)\to\infty$, as $x\to\infty$. Biernacki [Prace Mat.-Fiz. 40, 163–171 (1933)] has proved that if $y=y(x)\neq0$ is a solution of $y''+A(x)y=0$ and if $y(x)=0$ at $x=u_1, u_2, \cdots$ and $y'(x)=0$ at $x=v_1, v_2, \cdots$, where $u_1<v_1<u_2<v_2<\cdots$, then $(v_n-u_n)/(u_{n+1}-u_n)$ tends to $\frac{1}{2}$ if n, avoiding certain "exceptional" values, tends to ∞. The author constructs a function $A(x)$ and a solution $y=y(x)$ of the corresponding differential equation for which $(v_n-u_n)/(u_{n+1}-u_n)$ does not tend to $\frac{1}{2}$ if n tends to ∞ (without restriction). He states that in a similar manner one can obtain an example where $(u_{n+1}-u_n)/(u_n-u_{n-1})$ fails to tend to 1, as $n\to\infty$; this result is also complementary to one of Biernacki. *P. Hartman* (Baltimore, Md.).

Avakumović, V. G. Über die Nullstellen der Integrale nichtlinearer Differentialgleichungen zweiter Ordnung. Srpska Akad. Nauka. Zbornik Radova, Knj. 7. Matematički Institut, Knj. 1, 1–16 (1951). (Serbo-Croatian. German summary)

Assume that in the region D:

$$a\leqq x\leqq b,\quad |y|\leqq\tfrac{1}{2}M(x-a)(b-x)$$

the function $f(x)$ is continuous and satisfies $|f(x, y)|\leqq M$, $|f(x, y_1)-f(x, y_2)|\leqq\alpha|y_1-y_2|$ with $\alpha(b-a)^2<\pi^2$. Then the boundary value problem $y''=f(x, y)$, $y(a)=y(b)=0$, has exactly one solution in D and it can be found by successive approximations. This is a slight improvement over a result by Picard [Leçons sur quelques problèmes aux limites de la théorie des équations différentielles, Gauthier-Villars, Paris, 1930] inasmuch as the constant 8 is replaced by π^2 in the last condition, this being the best possible constant. It follows that if $f(x, 0)\equiv0$ and a, b are two consecutive zeros of $y''=f(x, y)$ then $b-a\geqq\pi/\alpha^{\frac{1}{2}}$. *M. Golomb.*

Reuter, G. E. H. On certain non-linear differential equations with almost periodic solutions. J. London Math. Soc. 26, 215–221 (1951).

The differential equation $\ddot{x}+kf(x)\dot{x}+g(x)=kp(t)$, $t>0$ is considered with f, g and p continuous, $xg(x)>0$ for $x\neq0$, $f(x)$ positive, and $p(t)$ and $P=\int_0^t p(t)dt$ uniformly bounded for $t>0$. Under quite general conditions on f and g it is known that there exist constants x_0 and v_0 independent of k such that every solution eventually satisfies $|x(t)|\leqq x_0$, $|\dot{x}(t)|\leqq v_0$. Let there further exist positive constants $a_1, \cdots, a_4$ and γ such that for $|x|\leqq x_0$, $a_1\leqq f(x)\leqq a_2$, $a_3\leqq g'(x)\leqq a_4$ and $|g''(x)|\leqq\gamma$. Let $k>k_0=v_0\gamma/a_1a_3$. Then if $p(t)$ is uniformly almost periodic (u.a.p.) all solutions of the differential equation converge to a u.a.p. solution $\alpha(t)$ with a u.a.p. derivative $\dot{\alpha}(t)$. Other results are given. *N. Levinson* (Cambridge, Mass.).

Langenhop, C. E. Note on Levinson's existence theorem for forced periodic solutions of a second order differential equation. J. Math. Physics 30, 36–39 (1951).

In a paper of the reviewer [same J. 22, 41–48 (1943); these Rev. 5, 66] the equation $\ddot{x}+f(x,\dot{x})\dot{x}+g(x)=e(t)$ is treated. The author relaxes the conditions on $g(x)$ by showing that it suffices for $\lim|g(x)|=\infty$ as $|x|\to\infty$, $xg(x)>0$ for $|x|>a>0$ and $\limsup g(x)/x=\infty$ as $x\to\infty$. *N. Levinson.*

*Helfenstein, Heinz. **Ueber eine spezielle Lamésche Differentialgleichung, mit Anwendung auf eine approximative Resonanzformel der Duffingschen Schwingungsgleichung.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1950. 67 pp.

The differential equation $y''\pm w_0^2y+\epsilon y^3=C\,\mathrm{cn}\,(wx,k)$, where w_0, ϵ, C, are given constants and $\mathrm{cn}\,(wx,k)$ is Jacobi's elliptic function, possesses for certain values of w and k a periodic solution of the form $y(x)=\mu\,\mathrm{cn}\,(wx,k)$. The corresponding variational equation is $y''=[6k\,\mathrm{sn}^2(x,k^2)+A]y$, where k and A are certain elementary functions of w_0, ϵ, C, w, k, μ. The solutions of this Lamé equation are simple combinations of θ-functions. These facts enable the author to make a detailed study of the resonance curves and the stability regions for these periodic solutions. For values of k that are not too large these results approximate the corresponding ones for Duffing's equation, for which such explicit calculations are not possible. The last chapter contains a method for the approximation of more general periodic functions by linear combinations of elliptic functions, in order to extend the computational method of the author to cases with a more general periodic forcing term.

W. Wasow (Los Angeles, Calif.).

Haag, Jules. Sur la synchronisation d'un oscillateur par une force sinusoïdale indépendante de la vitesse. C. R. Acad. Sci. Paris 233, 117–118 (1951).

The author points out an error in an argument by N. Minorsky [C. R. Acad. Sci. Paris, 232, 2179–2180 (1951); these Rev. 13, 38] to the effect that the differential equation $\ddot{x}+x+\epsilon x(A-Cx^2)\cos 2t=0$ possesses a stable periodic solution, if $AC>0$. By means of his own method [ibid. 232, 661–663 (1951); these Rev. 12, 707] he shows then that no such stable periodic solution exists, even for the more general differential equation $\ddot{x}+x+\epsilon H(x)\cos Nt=0$, where N is an integer.

W. Wasow (Los Angeles, Calif.).

Schaffner, Johannes S. Almost sinusoidal oscillations in nonlinear systems. I. Introduction—Simultaneous oscillations. University of Illinois Bulletin. Engineering Experiment Station Bulletin Series. no. 395, 64 pp. (1951).

An exposition is given of the method of "equivalent linearisation" [N. Kryloff and N. Bogoliuboff, Introduction to non-linear mechanics, Princeton University Press, 1943; these Rev. 4, 142] extended to systems with several degrees of freedom. Methods of calculation are developed for nonlinear elements with polynomial characteristic, and there is also a discussion of higher approximations. The last section deals with free oscillations, with particular emphasis on the possibility of "asynchronous" oscillations in systems with two degrees of freedom. *G. E. H. Reuter.*

Lyagina, L. S. The integral curves of the equation

$$y'=\frac{ax^2+bxy+cy^2}{dx^2+exy+fy^2}.$$

Uspehi Matem. Nauk (N.S.) 6, no. 2(42), 171–183 (1951). (Russian)

The configuration of the integral curves in the neighborhood of the origin can be reduced by means of an affine mapping to 16 different cases which are thoroughly discussed by the author.

J. L. Massera (Montevideo).

Kranjc, A. Sull'integrazione dell'equazione di Ritter Emden. Mem. Soc. Astr. Ital. (N.S.) 22, 127–130 (1951).

It is shown that the only group of transformations leaving the Emden equation $d^2y/dx^2+(2/x)dy/dx+y^n=0$ invariant is $X=tx$, $Y=t^{-2/(n-1)}y$ (for $n\neq 0,1$). *G. E. H. Reuter.*

Mitrinovitch, D. S. Troisième méthode d'intégration de l'équation de Neményi-Truesdell. Bull. Soc. Math. Phys. Macédoine 2, 17–24 (1951). (Serbo-Croatian. French summary)

The author continues his studies of the equation

$$F''/F+kf''/f=0,$$

k being a constant, with the objective of finding explicitly a pair of functions f and F which satisfy it. When success is achieved, $r=f(z)$ may be interpreted as the meridian curve of an elastic shell of revolution, for which the Neményi stress functions, yielding a complete solution of the problem of extensional stress subject to unsymmetrical load and support, are $F(z)$ with $k=n^2-1$, n being an integer. The present method begins with an arbitrary functional relation $F=T(f)$ and from it constructs the solution $\int df\exp\int df[fT''/(fT'+kT)]=Ax+B$, so that by means of two successive quadratures and one functional inversion a meridian function f is obtained. In comparing this method with his two previous ones [C. R. Acad. Sci. Paris 231, 327–328 (1950); these Rev. 12, 301; also the two papers reviewed below], the author concludes that the first is the most complicated, while the second is the simplest; that all three may be generalized so as to apply to equations of much more general type; but that only the third, in more general form, can be applied to the reviewer's equation $F''/F-(f'''/f''+f'/f)F'/F+n^2f''/f=0$, which is the governing equation for the corresponding extensional displacements. [For the author's treatment of this latter equation, see C. R. Acad. Sci. Paris 232, 681–683 (1951); these Rev. 12, 558]. The author has given elsewhere a more detailed treatment of his three methods in generalized form [see the third following review]. *C. Truesdell.*

Mitrinovitch, D. S. On an equation of Neményi and Truesdell. J. Washington Acad. Sci. 41, 129 (1951).

The author constructs a second method of inverse solution for the equation (*) $F''/F+(n^2-1)f''/f=0$. Let $\theta(z)$ be an arbitrary function, $\lambda=1-n^2$, $\Delta=(\lambda\theta^2+(\lambda-\lambda^2)\theta')^{\frac{1}{2}}$, $(\lambda-1)G=-\theta\pm\Delta$, $\lambda(\lambda-1)g=-\lambda\theta\pm\Delta$, the signs $\pm$ being associated. Further, let F and f be defined by $F=A\exp\int G\,dz$, $f=B\exp\int g\,dz$, where A and B are arbitrary constants. Then F and f satisfy (*).

C. Truesdell (Bloomington, Ind.).

Mitrinovitch, Dragoslav S. Sur l'équation différentielle d'un problème important de la théorie de l'élasticité. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 5, 22 pp. (1950). (Serbo-Croatian. French summary)

The author constructs an inverse method of solution for the equation (*) $F''/F+(n^2-1)f''/f=0$, to the integration of which Neményi and the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943); these Rev. **5**, 84] reduced the general equilibrium problem for the extensional stresses in a thin elastic shell of revolution with meridian $r=f(z)$. By applying a method of Darboux and Drach, the author is able to construct an infinite number of meridians f for which the solution F is obtainable by quadratures. He gives the following examples:

$$f=\frac{\sin^3 x}{\cos x}\left(\frac{1}{\sin x}\frac{d}{dx}\right)^3(Ae^z+Be^{-z}),\quad z=\tan x-x,$$

$$f=z\left\{A\left[C\left(\frac{1}{z}-\alpha\right)^{\frac{1}{2}}-1\right]\exp\left[C\left(\frac{1}{z}-\alpha\right)^{\frac{1}{2}}\right]+B\left[C\left(\frac{1}{z}-\alpha\right)^{\frac{1}{2}}+1\right]\exp\left[-C\left(\frac{1}{z}-\alpha\right)^{\frac{1}{2}}\right]\right\},$$

where A, B, C, α are arbitrary constants. A summary appeared earlier [C. R. Acad. Sci. Paris **231**, 327–328 (1950); these Rev. **12**, 301], and in a later work [see the following review] the author has generalized the present results.

C. Truesdell (Bloomington, Ind.).

Mitrinovitch, Dragoslav S. Sur une équation différentielle indéterminée. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 6, 16 pp. (1950). (Serbo-Croatian. French summary)

The author discusses three methods of solving the indeterminate differential equation $y''/y-z''/z=h=\text{const}$. In one method the equation is replaced by the equivalent system $y''=[\phi(x)+h]y$, $z''=\phi(x)z$ ($\phi(x)$ arbitrary), in another method solutions of the form $y=T(z)$ are sought. Admitting quadratures, the general solution is given as $y=\theta^{-\frac{1}{2}}\exp\{\frac{1}{2}\int(h/\theta+\theta)dx\}$, $z=\theta^{-\frac{1}{2}}\exp\{\frac{1}{2}\int(h/\theta-\theta)dx\}$, ($\theta(x)$ arbitrary).

M. Golomb (Lafayette, Ind.).

Charles, H. Sur la synthèse de la solution des équations de composition. Bull. Soc. Roy. Sci. Liège **20**, 100–106 (1951).

Using Laplace transforms the author observes the solutions of a few common problems in differential and integral equations in which the unit impulse function occurs either in the equation or in a boundary condition. With the aid of the convolution property of the transformation he also observes that those solutions serve as Green's functions for problems in which general functions appear in place of the impulse functions.

R. V. Churchill (Ann Arbor, Mich.).

Infeld, L., and Hull, T. E. The factorization method. Rev. Modern Physics **23**, 21–68 (1951).

This is a report on the "factorization" method for eigenvalue problems of wave mechanics. The method is set forth and applied to many classical equations of the type

$$(1)\qquad \frac{d^2y}{dx^2}+\{r(x,m)+\lambda\}y=0,$$

in which m is a positive integer (ordinarily introduced by a separation of variables) and λ a parameter. The interval of x is one at an end of which the equation is singular. The requirement of quadratic integrability for the solution is therefore of the nature of a boundary condition. It is ordinarily fulfilled only for certain λ eigenvalues, and certain associated values of m.

The method applies to equations (1) that can be "factorized." This means that functions $k(x, m)$ and $L(m)$ can be found, such that it can be written in each of the two forms

$$(2)\qquad \left[k(x,m+1)+\frac{d}{dx}\right]\left[k(x,m+1)-\frac{d}{dx}\right]y=\{\lambda-L(m+1)\}y,$$

$$(3)\qquad \left[k(x,m)-\frac{d}{dx}\right]\left[k(x,m)+\frac{d}{dx}\right]y=\{\lambda-L(m)\}y.$$

This is the case only for special types of functions $r(x, m)$. They are related with their associated functions $k(x, m)$ and $L(m)$ by equations that are obtained by identifying (2) and (3) with (1). There are six types in which $k(x, m)$ is a polynomial in m and $1/m$, with coefficients that are functions of x. Each of these types involves a number of constants, for specific values of which special cases are obtained. The method can be applied to a particular problem if that can be identified to be one of these special cases.

From (3) it can be drawn that

$$(4)\qquad \left[k(x,m+1)-\frac{d}{dx}\right]y(x,\lambda,m)=y(x,\lambda,m+1),$$

and from (2) that

$$(5)\qquad \left[k(x,m)+\frac{d}{dx}\right]y(x,\lambda,m)=y(x,\lambda,m-1).$$

Two cases are distinguished, according as $L(m)$ is an increasing or a decreasing function of m. In the former case, if $y(x, \lambda, m)$ is quadratically integrable, it can be shown that

$$(6)\qquad \int y^2(x,\lambda,p+1)dx=\{\lambda-L(p+1)\}\{\lambda-L(p)\}\cdots\times\{\lambda-L(m+1)\}\int y^2(x,\lambda,m)dx,$$

for $p=m+1, m+2, \cdots$. This is contradictory, since the right-hand member is negative for some sufficiently large p, if λ is not a value $L(l+1)$, with an integral l. Hence $\lambda_l=L(l+1)$ are the eigenvalues. From (6) then $y(x, \lambda_l, l+1)\equiv 0$, which by (4) gives $[k(x, l+1)-d/dx]y(x, \lambda_l, l)=0$. Hence

$$y(x,\lambda_l,l)=Ce^{\int k(x,\,l+1)\,dx},$$

and by the use of (5) the functions $y(x, \lambda_l, m)$ for $m=l-1, l-2, \cdots, 0$, are successively obtainable.

The discussion is much more complete than this summary can indicate. An extensive number of equations of physics are factorized and discussed. A bibliography is included.

R. E. Langer (Madison, Wis.).

Bopp, Fritz, und Haag, Rudolf. Über die Möglichkeit von Spinmodellen. Z. Naturforschung **5a**, 644–653 (1950).

The first section gives a short method of obtaining explicitly the angular dependent parts of the wave functions in the D_j representation of the rotation group. The method is well known. For integral j it is used by Condon and Shortley [The Theory of Atomic Spectra, Macmillan, New York, 1935, p. 50] and for arbitrary j by Infeld and his collaborators [e.g. §4.1–4.8 of the paper reviewed above]. The basic idea of the method, that an irreducible representation of a semi-simple group is completely determined by a dominant vector, is due to E. Cartan [Bull. Soc. Math.

France 41, 53–96 (1913)]. A model for an electron is proposed consisting of a rigid spherically symmetric electric charge. The classical Hamiltonian is written down and it is observed that if, after transition to quantum mechanics, the wave functions are taken from $D_{\frac{1}{2}}$ the Pauli matrices appear and spin has been incorporated in a classical model.

A. J. Coleman (Toronto, Ont.).

Mambriani, Antonio. Osservazioni su il classico teorema di confronto di Sturm. Rivista Mat. Univ. Parma **2**, 111–114 (1951).

The author deduces the comparison theorem for the equations $(\theta y')'-Qy=0$, $(\theta_1 z')'-Q_1 z=0$, with $\theta\geqq\theta_1>0$, $Q\geqq Q_1$, from the monotonic property of the "Picone auxiliary function" $\theta yy'-\theta_1 zz'(y/z)^2$. He next gives a comparison theorem regarding the zeros of z and of $y(\theta y'z-\theta_1 z'y)$. Finally he remarks that the conditions $\theta\geqq\theta_1$, $Q\geqq Q_1$, may be weakened to $k\theta\geqq\theta_1$, $kQ\geqq Q_1$, for positive constant k.

F. V. Atkinson (Ibadan).

***Weinstein, A. Quantitative methods in Sturm-Liouville theory.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 345–352. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

It is shown that the eigen-values of $v''-r(x)v+\lambda v=0$ with $v(0)=v(\pi)$ and $v'(0)=v'(\pi)$ are the zeros of

$$F(\lambda)=\sum_{n=1}^{\infty}\{u_n(\pi)-u_n(0)\}^2/(\lambda-\omega_n),$$

where u_n, ω_n are the eigen-functions and eigen-values of $u''-r(x)u+\lambda u=0$ with $u'(0)=u'(\pi)=0$. By this means the two sets of eigen-values are compared. *F. V. Atkinson.*

Sears, D. B. An expansion in eigenfunctions. Proc. London Math. Soc. (2) **53**, 396–421 (1951).

The paper is concerned with the differential equation

$$d^2y/dx^2+\{\lambda-q(x)\}y=0,$$

with real boundary conditions at $x=0$. The interval is $0\leqq x<\infty$, and $q(x)$ is real, continuous and $L(0,\infty)$. That the associated expansion formula for an arbitrary function $f(x)$ converges for $x>0$ is known if $f(x)$ is $L(0,\infty)$ and $L^2(0,\infty)$, and is of bounded variation in a neighborhood of x. It is shown here that the condition L^2 is unnecessary. Results are also obtained for the case L^2, and summability by Cesàro means is considered. The analysis is similar to that of Titchmarsh's book, Eigenfunction expansions associated with second-order differential equations [Oxford, 1946; these Rev. **8**, 458], and considerable references to the book are made. *R. E. Langer* (Madison, Wis.).

Sears, D. B. On the spectrum of a certain differential equation. J. London Math. Soc. **26**, 205–210 (1951).

The equation $y''+(\lambda-q(x))y=0$ is considered on $0<x\leqq b$ where $q(x)$ is real and continuous on this interval. Also $y(b)=\sin\alpha$, $y'(b)=-\cos\alpha$ for real α. The spectrum is shown to be discrete if $q(x)+1/(4x^2)\geqq A$ where A is a constant. Another condition for a discrete spectrum is $|q(x)|\leqq A+\delta/x^2$ for some δ, $0\leqq\delta<\frac{3}{4}$. Other conditions are given.

N. Levinson (Cambridge, Mass.).

Putnam, C. R. The comparison of spectra belonging to potentials with a bounded difference. Duke Math. J. **18**, 267–273 (1951).

The author establishes the following relation between the spectra $S_1(\alpha)$, $S_2(\alpha)$, of the boundary value problems on the half-line $0\leqq t<\infty$ determined by the respective differential equations

$$(1_k)\qquad (px')'+(\lambda+q_k(t))x=0 \qquad (k=1,2)$$

and the common boundary condition

$$(2)\qquad x(0)\cos\alpha+x'(0)\sin\alpha=0 \qquad (0\leqq\alpha<\pi).$$

Let p, q_1, q_2 be continuous functions on $0\leqq t<\infty$ satisfying

$$(3)\qquad p(t)>0,\quad |q_1(t)-q_2(t)|<\text{constant}<\infty$$

and such that (1_k) is in the Grenzpunktfall; then, if α is fixed and μ_1 denotes any point of $S_1(\alpha)$, there exists at least one point μ_2 of the set $S_2(\alpha)$ satisfying $|\mu_1-\mu_2|\leqq\sup|q_1(t)-q_2(t)|$. A similar relation between the sets S_1', S_2' of cluster points of $S_1(\alpha)$, $S_2(\alpha)$ is given in terms of the $\limsup_{t\to\infty}|q_1(t)-q_2(t)|$.

M. J. Gottlieb (Chicago, Ill.).

Gel'fand, I. M., and Levitan, B. M. On the determination of a differential equation by its spectral function. Doklady Akad. Nauk SSSR (N.S.) **77**, 557–560 (1951). (Russian)

The solution $y=\phi(x,\lambda)$ of the differential equation $y''+(\lambda-q(x))y=0$, with $\phi(0,\lambda)=1$, $\phi'(0,\lambda)=h$ has a spectral function $\rho(\lambda)$ such that any real $f(x)\in L^2(0,\infty)$ satisfies $\int_0^\infty f^2(x)dx=\int_{-\infty}^{\infty}E^2(\lambda)d\rho(\lambda)$ where

$$E(\lambda)=\text{l.i.m.}\int_0^n f(x)\phi(x,\lambda)dx$$

with $n\to\infty$. The determination of $q(x)$ from $\rho(\lambda)$ is considered and a number of sufficient conditions are stated for this to be the case. *N. Levinson* (Cambridge, Mass.).

Rellich, Franz. Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung. Math. Ann. **122**, 343–368 (1951).

This paper is concerned with singular boundary problems involving the differential equation

$$-(p(x)u')'+q(x)u=\lambda k(x)u, \qquad l<x<m,$$

where on the open interval $l<x<m$ ($l=-\infty$ or $m=+\infty$, or both, permitted) the real-valued functions $p(x)$, $q(x)$, $k(x)$ are such that $p(x)$ is of class C', $q(x)$, $k(x)$ are piecewise continuous, and $p(x)>0$, $k(x)>0$. When the operator $Au=k^{-1}\{-(pu')'+qu\}$ is bounded below in the sense that there is a finite constant a such that $\int_l^m k\bar{u}Au\,dx\geqq a\int_l^m k|u|^2dx$ on the set W of complex-valued $u(x)$ which are of class C', have piecewise continuous second derivatives on $l<x<m$, and $u(x)\equiv 0$ in some neighborhood of the individual endpoints $x=l$ and $x=m$, the author obtains the characterization of distinguished (ausgezeichnete) boundary conditions originally obtained by K. Friedrichs [Math. Ann. **112**, 1–23 (1935)] for a similar problem under stronger conditions upon the coefficients. In addition, two new characterizations of the distinguished boundary conditions are given. The first of these deals with the case of Au bounded below with regard to an individual end-point, and the determination of a function $\psi(x)$ such that the corresponding distinguished boundary condition becomes $u(x)/\psi(x)\to 0$ as x tends to the considered end-point. The second characterization states that in the class of possible boundary conditions $x=l$ and $x=m$ the distinguished boundary conditions are such that the operator Au is maximal in a certain sense.

W. T. Reid (Albuquerque, N. M.).

Povzner, A. On the differentiation of the spectral function of the Schrödinger equation. Doklady Akad. Nauk SSSR (N.S.) **79**, 193–196 (1951). (Russian)

Consider the differential operator $\mathfrak{L}=\Delta+c$, where Δ is Laplace's operator in three-dimensional space and $c=c(p)$ is a real continuous function. In the case when $\mathfrak{L}$ considered as an operator in L_2 has a self-adjoint closure, Carleman [Ark. Mat. Astr. Fys. **24B**, no. 11 (1934)] has shown that if E_λ is the corresponding resolution of the identity then $E_\lambda f(p)=\int\vartheta(p, q; \lambda)f(q)dq$ where $\vartheta(p, q; \lambda)$ is continuous and belongs to L_2 as a function of p or q. The author shows that there exists a function $\psi(p, q; \lambda)$ which is continuous in the pair p, q and a non-decreasing function $\tau(\lambda)$ such that $\int_{D_1}\int_{D_2}\int_I|\psi(p, q; \lambda)|d\tau(\lambda)dpdq<\infty$ when the regions D_1 and D_2 and the interval I are bounded,

$$\vartheta(p, q; \lambda)-\vartheta(p, q; \mu)=\int_\mu^\lambda\psi(p, q; \nu)d\tau(\nu),$$

and $\mathfrak{L}_p\psi(p, q; \lambda)+\lambda\psi(p, q; \lambda)=0$. Some properties of the resolvent of $\mathfrak{L}$ are also given (without proof).

L. Gårding (Lund).

Titchmarsh, E. C. Eigenfunction expansions associated with partial differential equations. Proc. London Math. Soc. (3) **1**, 1–27 (1951).

The author gives a detailed discussion of properties of the Green's function for the first boundary problem over a square domain for the partial differential equation

$$(*)\qquad \nabla^2\varphi+\{\lambda-q(x, y)\}\varphi=0.$$

He then proceeds by limiting processes to define a Green's function $G(x, y, \xi, \eta, \lambda)$ for (*) when the region of consideration is the entire (x, y)-plane, and derives an integral formula for arbitrary functions f such that f and $(q-\nabla^2)f$ belong to L^2, and satisfy at infinity the condition $f\partial G/\partial r-G\partial f/\partial r=o(1/r)$ for every complex value λ.

W. T. Reid (Albuquerque, N. M.).

★**Minakshisundaram, S. Expansion in eigenfunctions of the membrane problem.** Proceedings of the Symposium on Spectra Theory and Differential Problems, pp. 317–323. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

In this too brief note are stated convergence tests for a Fourier series $f(x, y)=\sum a_n w_n(x, y)$ in which $\{w_n(x, y)\}$ are the eigenfunctions and $\{\lambda_n\}$ are the eigenvalues (multiple ones occurring multiply) of the equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2+\lambda u=0$ with boundary values 0 for a domain with piecewise regular boundary. One theorem is that for $\sum\lambda_n a_n^2<\infty$ the series converges at all points of continuity of $f(x, y)$, and in general the series is summable spherically Riesz of order $k>1$. Another theorem is that, if the function qua function has a limit Riesz of order $p\geqq 0$, then the series converges Riesz of order $>\max(1, p+\frac{1}{2})$; and if the series is summable $k\geqq 0$ then the function has a limit for $p>R+\frac{1}{2}$. *S. Bochner.*

★**Minakshisundaram, S. Lattice point problems and eigenvalue problems.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 325–332. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

The author more or less restates some results previously given by himself, and by himself and Å. Pleijel, on the behavior of $\sum w_n(x)w_n(y)/\lambda_n^s$ as a function in s, if $\{w_n(x)\}$ and $\{\lambda_n\}$ are eigenfunctions and eigenvalues pertaining to $\sum_{j=1}^k\partial^2u/\partial x_j^2+\lambda u=0$, with boundary values or normal derivatives vanishing for a Euclidean domain with regular boundary. *S. Bochner* (Princeton, N. J.).

Takeyama, Hisao. On the orthogonalities of the normal functions in the lateral vibrations of plates. J. Jap. Soc. Appl. Mech. **4**, 63–66 (1951).

Although his results have been established by other methods, the author considers that it is of interest to prove the orthogonality of the normal functions of a vibrating plate by direct transformations of the appropriate integral. This is done, using orthogonal curvilinear coordinates, for the case of simply supported edges and that of free edges.

H. W. March (Madison, Wis.).

★**Petrovskiĭ, I. G. Lekcii ob uravneniyah c častnymi proizvodnymi. [Lectures on Partial Differential Equations].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 303 pp.

The present book is a concise and lucidly written introduction to the subject of partial differential equations. Important results are carefully stated in the form of theorems, bringing out the relevant hypotheses in each case. The proofs are presented clearly and directly. Chapter I, entitled "Introduction. Classification of equations" deals with Cauchy's problem, S. Kowalewsky's existence theorem, characteristics, E. Holmgren's uniqueness proof for Cauchy's problem, canonical forms for second order linear partial differential equations in one unknown function of two independent variables, and canonical forms for systems of linear first order partial differential equations in two independent variables. Chapter II, entitled "Hyperbolic equations", is divided into two parts: (a) Cauchy's problem in the domain of non-analytic functions and (b) vibrations. Part (a) deals with the "correct posing" of Cauchy's problem, Cauchy's problem for the wave equation in one, two, and three space-dimensions and for hyperbolic systems in two independent variables, Lorentz transformations, mathematical foundations of special relativity. Part (b) deals with vibration problems, the so-called "mixed" problems for the wave equation, and specially Fourier's method (expansion in terms of particular solutions obtained by the method of separation of variables) for the vibrating string equation. Two proofs of the needed eigenfunction expansions are given, the first is variational, and the second by means of integral equations. Chapter III, entitled "Elliptic equations", deals with Laplace's equation, potential theory, solution of Dirichlet's problem for a circle by Poisson's integral. The uniqueness of the solution of the Dirichlet problem is proved by an elementary method (not involving Green's theorem) due to I. I. Privalov [Mat. Sbornik (1) **32**, 464–469 (1925)] and the existence of the solution by means of the Poincaré-Perron method of sub- and super-harmonic functions. The difference equation method for the approximate solution of the Dirichlet problem is also considered. Parabolic equations are discussed very briefly in chapter IV. An interesting feature of the book is the presence, at the end of each of the last three chapters, of a brief but informative survey of related known results. *J. B. Diaz.*

Germay, R. H. J. Sur les équations récurrentes aux dérivées partielles du premier ordre, de forme résolue par rapport à une dérivée. Bull. Soc. Roy. Sci. Liège **19**, 507–513 (1950).

The author considers the sequence of partial differential equations of the first order solved for the derivative with

respect to x_1,

$$(1)\quad \frac{\partial u_i}{\partial x_1}=f_i\Big(x_1,\cdots,x_n;u_i,\cdots,u_{i+r};\frac{\partial u_i}{\partial x_2},\cdots,\frac{\partial u_i}{\partial x_n};\cdots;\frac{\partial u_{i+r+1}}{\partial x_2},\cdots,\frac{\partial u_{i+r}}{\partial x_n}\Big)\qquad(i=1,2,\cdots)$$

and a sequence of functions (2) $\varphi_i(x_2,\cdots,x_n)$ $(i=1,2,\cdots)$, arbitrary but supposed to satisfy

$$(3)\quad \varphi_m(x_2^0,\cdots,x_n^0)=v_1^0,\cdots,\varphi_{m+r}(x_2^0,\cdots,x_n^0)=v_{r+1}^0,\quad \frac{\partial\varphi_m(x_2^0,\cdots,x_n^0)}{\partial x_2}=v_{1,2}^0,\cdots\quad(m=1,2,\cdots)$$

which are the initial conditions of the equations (1). He asserts then that these equations admit a system of holomorphic integrals $u_k(x_1,\cdots,x_n)$ which for $x_1=x_1^0$ reduce to the functions $\varphi_k(x_2,\cdots,x_n)$. In order to prove it, he sets $\partial u_m/\partial x_n=p_{m,n}$ and goes from (1) to a system of which the first members are $\partial p_{m,1}/\partial x_1$, $\partial p_{m,2}/\partial x_1,\cdots$ $(m=1,2,\cdots)$. This system admits holomorphic integrals satisfying the initial conditions

$$U_m(x_1^0,x_2,\cdots,x_n)=\varphi_m(x_2,\cdots,x_n),\cdots,$$
$$U_{m+r}(x_1^0,x_2,\cdots,x_n)=\varphi_{m+r}(x_2,\cdots,x_n);$$
$$\Pi_{m,2}(x_1^0,x_2,\cdots,x_n)=\partial\varphi_m/\partial x_2,\cdots,$$
$$\partial U_m/\partial x_1=f_m(x_1^0,x_2,\cdots).$$

Now it follows from $(\partial/\partial x_1)(\partial U_m/\partial x_2)=\partial\Pi_{m,2}/\partial x_1$, and from the initial conditions, that

$$\partial U_m(x_1,\cdots,x_n)/\partial x_2=\Pi_{m,2}(x_1,\cdots,x_n),\cdots.$$

In this way it is proved that the holomorphic functions $U_m,\cdots,U_{m+r}$, which satisfy the given initial conditions are the only solutions of equations (1). *P. Zervos* (Athens).

Ascoli, Guido. L'isotropia analitica e le sue applicazioni. Rend. Sem. Mat. Fis. Milano **20** (1949), 13–25 (1950).

This is a semi-expository paper written to clarify and complement an earlier contribution of the author [Pont. Acad. Sci. Comment. **7**, 207–281 (1943); these Rev. **10**, 123]. The method, linear isotropic function and operator systems, is applied to the two-dimensional case to show its power in synthesizing known results, particularly in potential theory. Indications are given of its principal utility, the generalization of classical theory to higher dimensions through the medium of the corresponding rotation group. *J. L. Vanderslice* (College Park, Md.).

Schwartz, Laurent. Les équations d'évolution liées au produit de composition. Ann. Inst. Fourier Grenoble **2** (1950), 19–49 (1951).

The author studies certain partial- and integro-differential equations from the viewpoint of his theory of distributions [Théorie des distributions, I, II, Actualités Sci. Ind., nos. 1091, 1122, Hermann, Paris, 1950 and 1951; these Rev. **12**, 31, 833] in partial extension and simplification of work of Petrowsky [Bull. Math. Univ. Moscou **1**, no. 7 (1938)] dealing with the case of functions. Various questions of existence, uniqueness, and dependence on initial data are treated. The equations considered are of the form

$$\frac{d}{dt}U_x(t)+A_x(t)*U_x(t)=B_x(t)$$

where t ranges over the interval $[a,b]$; x ranges over euclidean n-space E_n; for each t, $U_x(t)$ is a distribution on E_n (or N-vector of such) which is to be continuously differentiable as a function of t; $A_x(t)$ and $B_x(t)$ are given continuous distribution-valued functions of t ($A_x(t)$ being a $N\times N$ matrix of distributions actually); and the $*$ indicates convolution on E_n. The main tool in the present investigation is the resolvent matrix $\mathfrak{R}(y,t,\tau)$; here y is the variable dual to x; t and τ range over $[a,b]$, $\mathfrak{R}(y,\tau,\tau)$ is the identity matrix, and $\partial\mathfrak{R}(y,t,\tau)/\partial t+\mathfrak{A}(y,t)\mathfrak{R}(y,t,\tau)=0$, where $\mathfrak{A}(y,t)$ is the spatial Fourier transform of $A(x,t)$. The existence of such a matrix follows along more or less classical lines. Its spatial Fourier transform $R_x(t,\tau)$ has the property that $R_x(t_3,t_2)*R_x(t_2,t_1)=R_x(t_3,t_1)$ for $t_3>t_2>t_1$, and can be interpreted as the elementary wave for the system at the time τ and origin in space. The elementary kernel (essentially the elementary solution in the case of a differential equation) is $Y(t-\tau)R_{x-\xi}(t,\tau)$ where $Y(t)=1$ for $t>0$ and $=0$ for $t<0$.

The Cauchy initial value problem (i.e. $U_x(a)$ a given distribution) has at most one solution in the space $\mathcal{S}_x'$ of temperate distributions (i.e. $U_x(t)\in\mathcal{S}_x'$ for each t), with suitable restrictions on the distribution spaces in which the coefficients lie; if the solution is not required to be temperate, a well-known example shows that it need not be unique. With the same restrictions, the homogeneous Cauchy problem has a solution if and only if each partial derivative with respect to y of the resolvent matrix $\mathfrak{R}(y,t,a)$ is majorized by a polynomial in y independent of t. The initial value problem is said to be uniformly well posed (u.w.p.) in $[a,b]$ for a space $\mathcal{K}_x$ of distributions if for each $t_0\in[a,b]$ the initial value problem on $[t_0,b]$ has a solution in $\mathcal{K}_x$ that depends continuously in t, t_0, and $U_x(t_0)$. For the homogeneous initial problem to be u.w.p. in $[a,b]$ for $\mathcal{S}_x'$, it is necessary and sufficient that $\mathfrak{R}(y,t,\tau)$ be majorized by a polynomial in y independently of t and τ, for $a\leqq\tau\leqq t\leqq b$. Whenever the homogeneous Cauchy problem is u.w.p. in $[a,b]$ for $\mathcal{S}_x'$, the same is true of the non-homogeneous problem.

A result covering functions as well as distributions is as follows. Let $\mathcal{E}$ be the space of infinitely differentiable functions on E_n and $\mathfrak{D}$ the subset of all elements vanishing outside compact sets; let $\mathcal{E}'$, $\mathfrak{D}'$ be the duals of these spaces after they are topologized in certain fashions. In order that the initial problem be u.w.p. on $[a,b]$ for one of the spaces $\mathfrak{D}$, $\mathcal{E}$, $\mathfrak{D}'$, $\mathcal{E}'$, it is necessary and sufficient that it be such for $\mathcal{E}'$, and that the distribution $R_x(t,\tau)$ be in $\mathcal{E}_x'$ and have its support inside a fixed compact set (in E_n). The original equation is called hyperbolic by the author if the initial problem is u.w.p. in $\mathcal{E}'$. A partial differential equation in which no derivatives higher than the first appear is hyperbolic provided the homogeneous initial problem is u.w.p. in $[a,b]$ for $\mathcal{E}'$.

The initial problem may have a solution or be u.w.p. (for a given equation) without the same being true of the terminal value problem; this is true for example in the case of the heat equation. If, however, the initial problem is u.w.p. in $[a,b]$, then the terminal value problem for the adjoint equation is u.w.p. in $[a,b]$. When A_x is independent of time, solution of a u.w.p. initial problem leads to a semigroup which can be extended to a group when the terminal problem is also u.w.p. The paper closes with a discussion of 5 examples, including the heat and homogeneous wave equations. *I. E. Segal* (Princeton, N. J.).

Garnir, H. G. Détermination de la distribution résolvante de certains opérateurs d'évolution décomposables. Bull. Soc. Roy. Sci. Liège **20**, 96–99 (1951).

The resolvent distribution or inverse of the operator $E_1\cdots E_p$ is determined, where $E_i=\sum_{j=1}^{n+1}\alpha_{ij}\partial/\partial x_j+k_i$, the

α_{ij} and k_i being constants such that $\alpha_{i,n+1}>0$ and the vectors $\beta_i=(\alpha_{i1}, \cdots, \alpha_{i,n+1})$ are linearly independent.

I. E. Segal (Princeton, N. J.).

Garnir, H. G. Sur les distributions résolvantes des opérateurs de la physique mathématique. Bull. Soc. Roy. Sci. Liège **20**, 174–194 (1951).

Dans une première partie, l'auteur donne les solutions élémentaires d'opérateurs D elliptiques classiques $(\Delta-k^2)$ et $(\Delta-k^2)(\Delta-l^2)$ dans l'espace à n dimensions $(n=1, 2, 3)$. [Ces résultats sont connus. Mais l'auteur donne la définition rigoureuse de la solution élémentaire, une distribution T telle que $DT=\delta$, mesure de Dirac. Celà lui permet d'utiliser des calculs algébriques simples; par exemple si E_k est la solution élémentaire de $\Delta-k^2$, celle de $(\Delta-k^2)(\Delta-l^2)$ est $(E_k-E_l)/(k^2-l^2)$.] Dans la deuxième partie l'auteur définit une transformation de Laplace d'une distribution G des variables x (spatiale à n dimensions) et t (temps), par rapport à t seul (utilisation du théorème d'approximation de Weierstrass pour l'unicité). L'auteur ne se place pas dans un cadre assez général (la solution élémentaire de l'équation des ondes à 4 variables d'espace-temps échapperait à ce procédé), mais la modification à apporter pour généraliser est très faible. Cette méthode sert à transformer un problème d'équation hyperbolique à un problème d'équation elliptique; les équations d'ondes classiques sont justiciables d'un tel procédé.

L. Schwartz (Nancy).

Rubinštein, L. I. On the uniqueness of solution of the homogeneous problem of Stefan in the case of a single-phase initial condition of the heat conducting medium. Doklady Akad. Nauk SSSR (N.S.) **79**, 45–47 (1951). (Russian)

The following theorem is proved: Let $u_{ij}(x, t)$, $y_j(t)$ be two systems of solutions of the problem of Stefan:

$$\frac{\partial^2 u_{1j}}{\partial x^2}=\frac{\partial u_{1j}}{\partial t} \quad (0<x<y_j(t)); \qquad a^2\frac{\partial^2 u_{2j}}{\partial x^2}=\frac{\partial u_{2j}}{\partial t} \quad (y_j(t)<x<1);$$

$$u_{1j}(0, t)=f_1(t)\leqq 0; \quad u_{ij}(y_j(t), t)=0; \quad u_{2j}(1, t)=f_2(t)\geqq 0;$$

$$u_{ij}(x, 0)=\varphi_i(x); \quad (\varphi_1(x)\leqq 0; \ \varphi_2(x)\geqq 0);$$

$$\frac{dy_j}{dt}=\frac{\partial u_{1j}}{\partial x}-\frac{\partial u_{2j}}{\partial x}\bigg|_{x=y_j(t)}; \quad y_j(0)=l\subset[0, 1] \quad (j=1, 2).$$

Then $t=0$ appears as an accumulation point of the zeros of the difference $z(t)\equiv y_1(t)-y_2(t)$. The proof is based on the principle of the maximum for subparabolic functions and from this proof it follows that there does not exist two systems of solutions of the problem of Stefan in which the boundary separating the phases are analytic functions of t.

C. G. Maple (Washington, D. C.).

Rubinštein, L. I. On the propagation of heat in a two-phase system having cylindrical symmetry. Doklady Akad. Nauk SSSR (N.S.) **79**, 945–948 (1951). (Russian)

In this note the cylindrical problem of Stefan is considered under the hypothesis that there exists an axial symmetry. The method of solution and the proof of uniqueness are essentially the same as those for the linear problem of Stefan as given by the author in an earlier paper [same Doklady **58**, 217–220 (1947); these Rev. **9**, 287].

H. P. Thielman (Ames, Iowa).

Evans, G. W., II. A note on the existence of a solution to a problem of Stefan. Quart. Appl. Math. **9**, 185–193 (1951).

The author proves the existence and uniqueness of the solution of the problem of Stefan: $u_{xx}=u_t$, $0<x<x(t)$, with the boundary conditions $u=0$ for $x=x(t)$, $\dot{x}(t)=-u_x[x(t), t]$, $x(0)=0$, $u_x(0, t)=-1$. The method of proof consists of applying an iteration process to the equation

$$x(t)=t-\int_0^{x(t)} u(x, t)dx,$$

which is derived by evaluating $\int_0^t\int_0^{x(t)}(u_{xx}-u_t)dxdt=0$ subject to the above boundary conditions. The author makes extensive use of a theorem due to M. Picone [Math. Ann. **101**, 701–712 (1929)], later generalized by L. Nirenberg (unpublished) which essentially asserts that if a solution of the heat equation assumes its maximum or minimum at an interior point of its region of definition, then it is constant over a certain subregion.

C. G. Maple.

Lineĭkin, P. S. On the equations of heat convection. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 433–438 (1951). (Russian)

The author seeks to justify a "small parameter method" for the two-dimensional problem of stationary heat convection in a homogeneous liquid contained in a given region with prescribed temperatures on the boundary. The density is taken to depend linearly upon the temperature. The equations of heat-conduction and hydromechanics are reduced to the system

$$\nabla^4\psi=-\frac{\partial T}{\partial x}+g\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)},$$

$$\nabla^2 T=\frac{\partial(\psi, T)}{\partial(x, y)}-\epsilon D(\psi),$$

where

$$D(\psi)=4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right)^2.$$

Conditions are then found for the convergence of an iterative process using Green's functions for ∇^2 and ∇^4. The method itself appears to have been used already by the author [Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **11**, 103–126 (1947); these Rev. **8**, 611].

F. V. Atkinson.

Conti, Roberto. Sul problema di Cauchy per l'equazione $y^{2a}k^2(x, y)z_{xx}-z_{yy}=f(x, y, z, z_x, z_y)$, con i dati sulla linea parabolica. Ann. Mat. Pura Appl. (4) **31**, 303–326 (1950).

This paper is concerned with the Cauchy problem for the equation $y^{2a}k^2(x, y)z_{xx}-z_{yy}=f(x, y, z, z_x, z_y)$ with data prescribed on the parabolic line: $z(x, 0)=\varphi(x)$, $z_y(x, 0)=\psi(x)$, $a\leqq x\leqq b$. The results given here are generalisations of those obtained by Berezin [see Mat. Sbornik N.S. **24**(66), 301–320 (1949); these Rev. **11**, 112] for the case of an f, which is linear in z, z_x, z_y. The constant α is assumed to lie between 0 and 1. The functions $k(x, y)$, $f(x, y, z, p, q)$, $\varphi(x)$, $\psi(x)$ shall be such that in suitable regions k is positive, k_x, k_y, k_{xx}, k_{xy}, f, are continuous, $\varphi''(x)$ and $\psi'(x)$ satisfy a Lipschitz condition with respect to x, and f_x, f_z, f_p, f_q satisfy Lipschitz conditions with respect to z, p, q. It is shown that under these assumptions there exists a solution of the Cauchy problem in a region of the upper half-plane bounded by characteristics through the points $(a, 0)$, $(b, 0)$, and by the line $y=0$ and a suitable line $y=y_1$. The solution is uniquely determined, and together with its first derivatives depends

continuously on the functions $\varphi(x)$, $\psi(x)$ and their first derivatives. The method of proof consists in a reduction of the Cauchy problem to a system of 3 non-linear integral equations of the type of Volterra, which are solved by iteration. *F. John* (New Rochelle, N. Y.).

Martin, Monroe H. Riemann's method and the problem of Cauchy. Bull. Amer. Math. Soc. **57**, 238–249 (1951).

The author recalls first of all the essential points in Riemann's method of solving the problem of Cauchy for the partial differential equation

$$u_{xy}-a(u_x-u_y)=0, \quad a=a(x, y),$$

as follows: (a) the introduction of the characteristics as coordinate lines; (b) the construction of a line integral $I=\int(Bdx-Ady)$, which vanishes around closed paths, where (c) A and B are certain bilinear forms in u, u_x, u_y and v, v_x, v_y and (d) v (Riemann's function) is a properly chosen two-parameter family of solutions of a second linear partial differential equation, the adjoint equation. An extension of Riemann's method has been given by H. Lewy [Nachr. Ges. Wiss. Göttingen **1928**, 118–123]. This method employs three "Riemann functions" and neither makes use of characteristic coordinates, nor of an "associate equation". In seeking an alternative approach to the solution of Cauchy's problem the author constructs the line integral $I=\int\{\lambda^{-1}u_xv_xdx+\mu^{-1}u_yv_ydy\}$ for the partial differential equation

$$\mathfrak{L}(u)=u_{xy}-au_x-bu_y=0, \quad a=a(x, y), \quad b=b(x, y),$$

which vanishes around closed paths in the interior of a domain within which u, v are regular solutions of $\mathfrak{L}(u)=0$, $\Lambda(v)=v_{xy}-\alpha v_x-\beta v_y=0$, $(\alpha=\alpha(x, y), \beta=\beta(x, y), \lambda=\Phi_x, \mu=\Phi_y, \mathfrak{L}(\Phi)=0, \alpha=\Phi_x^{-1}\Phi_yb, \beta=\Phi_x\Phi_y^{-1}a)$. Guided by the procedure in Riemann's method and evaluating this integral around a closed path, the author obtains the theorem: The solution to Cauchy's problem for the partial differential equation

$$\mathfrak{L}(u)=u_{xy}-au_x-bu_y=0$$

with initial data prescribed along the arc C (as in Riemann's method) is given by

$$u(P)=\tfrac{1}{2}[u(X)+u(Y)]+\frac{1}{2}\int_X^Y\{\Phi_x^{-1}u_xv_xdx+\Phi_y^{-1}u_yv_ydy\},$$

where $\Phi=\Phi(x, y)$ is any regular particular solution of $\mathfrak{L}(u)=0$ for which $\Phi_x\neq 0$, $\Phi_y\neq 0$ and $v=v(x, y; \bar{x}, \bar{y})$, a resolvent of $\mathfrak{L}(u)=0$, is a two-parameter family of solutions of the associate equation

$$\Lambda(v)=v_{xy}-b\Phi_x^{-1}\Phi_yv_x-a\Phi_y^{-1}\Phi_xv_y=0$$

subject to the initial conditions $v_x=\Phi_x$ on $y=\bar{y}$, $v_y=-\Phi_y$ on $x=\bar{x}$, upon the characteristics through P. Applications are made to the equation for cylindrical waves. It turns out that axially symmetric solutions are governed by Euler's partial differential equation $2(\alpha-\beta)u_{\alpha\beta}-(u_\alpha-u_\beta)=0$.

M. Pinl (Dacca).

Charles, H. Sur l'équation des ondes dans le domaine héréditaire. Bull. Soc. Roy. Sci. Liège **20**, 195–198 (1951).

A modification of the wave equation that allows for certain hereditary properties of the medium has the form $U_{tt}-U_{xx}=\int_0^t K(t-\tau)U_{xx}(x, \tau)d\tau$, where the function $K(t)$ depends on the medium. The author determines the Laplace transform $u(x, s)$ of $U(x, t)$ in terms of the transform of $K(t)$ using the boundary conditions $U(x, 0)=U_t(x, 0)=0$, $U(l, t)=0$, $U(0, t)=\delta(t)$, where $\delta(t)$ is the unit impulse function. He determines $U(x, t)$ explicitly when $K(t)=\lambda^2t$, where λ is a constant. He also observes that when $K(t)=-\lambda\sin\lambda t$ the function $U(x, t)$ is the solution of the corresponding problem with linear damping, obtained by replacing the right-hand member of the modified wave equation by $-\lambda^2U(x, t)$. *R. V. Churchill* (Ann Arbor, Mich.).

Bers, Lipman. Isolated singularities of minimal surfaces. Ann. of Math. (2) **53**, 364–386 (1951).

The nonparametric differential equation of minimal surfaces may be considered the most accessible significant example revealing typical qualities of solutions of nonlinear partial differential equations. With a view to such a general objective, the author has studied singularities, branch-points and behavior in the large of minimal surfaces.

A solution $\phi(x, y)$ of the equation of minimal surfaces is said to possess at $z_0=x_0+iy_0$ an isolated singularity if it can be continued analytically along every path in the domain $0<|z-z_0|<\epsilon$. If this continuation leads to m determinations for the gradient $\phi_x-i\phi_y$, then z_0 is called a branch-point of order $m-1$. The present paper gives a complete description of all possible branch-points of finite order. The first result asserts that as $x+iy\to z_0$, $\phi_x-i\phi_y$ approaches a limit w_0. If $\phi(x, y)$ is not a linear function, then $\phi_x-i\phi_y\neq w_0$ at all points distinct from and sufficiently close to z_0. The integer

$$n=\frac{1}{2\pi}\int_C d\arg[\phi_x-i\phi_y-w_0]$$

where C is a circle $|z-z_0|=\epsilon'<\epsilon$ is called the index of the branch-point. If $n\geqq 0$ the branch-point is called ordinary. In this case w_0 is finite, and $w_0=0$ if and only if $n>0$. The function $\phi(x, y)$ behaves at z_0 like a harmonic function having an algebraic singularity, it is asymptotically equal to $\Re\{A+B(z-z_0)^{1+(n/m)}\}$. If $n<0$, then $n\geqq -m$ and $w_0=\infty$. Such a branch-point is called polar. There exists a finite number of topologically distinct types of polar branch-points of given order and index. Except for the case $n=-m$ these singularities are topologically different from those of harmonic functions. From the description of all finite branch-points it follows that at a finite point a single-valued solution of the equation of minimal surfaces can not have a non-removable isolated singularity.

A similar discussion is carried out for the case when the isolated singularity is at the point at infinity. All branch-points of finite order located at $z=\infty$ are ordinary, that is $n\geqq 0$ and $\phi_x-i\phi_y$ approaches a finite limit as $x+iy\to\infty$. The function $\phi(x, y)$ is asymptotically equal to $\Re\{A+Bz^{1-(n/m)}+C\log z\}$. If $\phi(x, y)$ is single-valued, then $\phi=ax+by+c\log(x^2+y^2)+O(1)$.

The paper also contains a new proof of S. Bernstein's well known theorem which states that a solution $\phi(x, y)$ regular at all finite points is a linear function. The proofs are based on a special form of the Monge-Weierstrass representation of minimal surfaces and on several new theorems concerning mapping by a pair of nonconjugate harmonic functions.

R. Courant (New York, N. Y.).

Bers, Lipman. Abelian minimal surfaces. J. Analyse Math. **1**, 43–58 (1951).

This paper continues the function-theoretical investigation of the class of functions satisfying the equation of minimal surfaces. In this class there are nontrivial functions

corresponding to entire functions or everywhere meromorphic functions. This follows from S. Bernstein's well known theorem and from the author's theorem on removable singularities. There are, however, infinitely many solutions of the equation considered corresponding to algebraic functions and their integrals.

A solution $\phi(x, y)$ of the equation of minimal surfaces is said to define an Abelian minimal surface if (a) $\phi_x - i\phi_y$ is a finitely-many valued function, (b) $\phi(x, y)$ can be continued analytically along every finite path in the plane which avoids a finite number of excluded points. From the results of another paper [see the preceding review] it follows that (c) $\phi_x - i\phi_y$ has a limit along every polygonal path leading to one of the excluded points or to the point at infinity. If $\phi(x, y)$ were a solution of the Laplace equation, conditions (a), (b), (c) would insure that it is the real part of an Abelian integral, hence the name Abelian minimal surface. Over the excluded points and at the point at infinity an Abelian minimal surface possesses branch-points of the kind described in the author's paper reviewed above. If the surface is not a plane, at least one of the branch-points must be polar, the sum of the indices of all branch-points is zero, and $\phi_x - i\phi_y$ takes on every value the same number of times.

To every Abelian minimal surface S there belongs a uniquely determined real plane algebraic curve C such that S admits an explicit parametric representation in terms of certain Abelian integrals attached to C. The real branches of C are in a one-to-one correspondence with the polar branch-points of S. The Riemann surface of C is always ortho-symmetric. Conversely, to every ortho-symmetric Riemann surface there belong infinitely many algebraic curves defining Abelian minimal surfaces. *R. Courant.*

Bers, Lipman. Boundary value problems for minimal surfaces with singularities at infinity. Trans. Amer. Math. Soc. 70, 465–491 (1951).

This paper deals with a class of boundary value problems of which the following is typical. Let P be a given simple closed curve in the plane, z_T a point on P. It is required to find a solution $\phi(x, y)$ of the equation of minimal surfaces defined in the domain exterior to P and such that (1) ϕ_x, ϕ_y are single-valued and $\max(\phi_x^2+\phi_y^2)=m^2=$given number; (2) ϕ_x approaches a positive limit, $\phi_y \to 0$ as $x^2+y^2 \to \infty$; (3) $\phi_x^2+\phi_y^2=0$ at z_T; (4) $\partial\phi/\partial n=0$ on P. If the equation of minimal surfaces is interpreted as the potential equation of a "Chaplygin gas", this problem requires one to find a flow past P having a prescribed direction at infinity, attaining a prescribed maximum speed, and satisfying at z_T the Kutta-Joukowsky condition. The other boundary value problems considered admit similar interpretations, or are obtained by replacing (4) by the boundary condition: $\phi=0$ on P. Thus one problem consists of finding a single-valued solution $\phi(x, y)$ defined in the domain exterior to P, vanishing on P, and such that $\max(\phi_x^2+\phi_y^2)=m^2$, and $\phi_x^2+\phi_y^2 \to 0$ as $x^2+y^2 \to \infty$. The profile P is assumed to consist of a finite number of smooth arcs along each of which the slope of the tangent considered as a function of arc-length is absolutely continuous and satisfies a Hölder-condition. The profile may have intruding corners, but no protruding corners, except at points at which a Kutta-Joukowski condition is imposed. Also, P is to be "essentially convex", that is, the tangent to P must be turned in the clockwise direction by less than π if P is traversed once in the positive direction.

Using a special form of the Monge-Weierstrass representation of minimal surfaces, each boundary value problem is transformed into a mapping problem (which contains the conformal mapping problem as a limiting case, $m=0$). The domain $|\zeta|>1$ is to be mapped onto the domain exterior to P by two harmonic functions satisfying certain auxiliary conditions. The mapping problem is then transformed into a nonlinear integro-differential equation for the function expressing the mapping of $|\zeta|=1$ onto P. Finally the solvability of the integro-differential equation is established by the topological method of Leray and Schauder. This part of the existence proof leans heavily on the theory of conjugate functions. *R. Courant* (New York, N. Y.).

Schade, Th. Neuartige Behandlung der Poissonschen und der inhomogenen Bipotentialgleichung bei rechteckigen Bereichen mit Anwendung auf Probleme der Torsion und der Plattenbiegung. Ing.-Arch. 19, 118–127 (1951).

R. Grammel has introduced [Ing.-Arch. 16, 188–200 (1948); 18, 250–254 (1950); these Rev. 10, 38; 12, 444] generalized trigonometric functions $x=\cos_n v$, $y=\sin_n v$ such that (x, y) can be considered as coordinates of a point on the curve $x^n+y^n=1$ (n even) and v as the double of the area between the positive x-axis, the curve and the radius vector. The present paper shows the application of the new functions to the solution of the boundary value problems, for rectangular regions, of the Poisson equation and the equation $\nabla^2\nabla^2 F(x, y)=f(x, y)$ (∇^2=Laplacian operator). These differential equations are transformed to an oblique coordinate system with the lines $v=$const. and $\rho=$const. as coordinate lines, ρ being defined by the equation $x^n+\alpha^n y^n=\rho^n$ where α is the length/width ratio of the rectangle. A set of solutions of the homogeneous equations in terms of the new coordinates for the limiting case $n \to \infty$ is deduced, and the solution is expressed in terms of these "new harmonics". At the diagonal lines discontinuities in the higher derivatives are, in general, unavoidable, while at all other inside points the differential equation is strictly satisfied and the boundary condition can be approximated as closely as desired.
P. F. Neményi (Washington, D. C.).

Difference Equations, Special Functional Equations

✓ ***Milne-Thomson, L. M. The Calculus of Finite Differences.** Macmillan and Co., Ltd., London, 1951. xxiii +558 pp. $4.50.
Reprinted from the first edition of 1933.

Hahn, Wolfgang. Über die Reduzibilität einer speziellen geometrischen Differenzengleichung. Math. Nachr. 5, 347–354 (1951).
Let $f(x)$ satisfy the functional equation of order n

$$(1) \quad \sum_{r=0}^{n}(B_r - A_r x)f(xq^r)=0, \quad A_0=B_0=1, \quad \sum_{r=0}^{n}B_r=0$$

in which q is a fixed number. The author puts

$$\sum_{r=0}^{n}A_r z^r=\prod_{s=1}^{n}(1-a_s z), \quad \sum_{r=0}^{n}B_r(zq)^r=\prod_{s=1}^{n}(1-b_s z), \quad b_n=q$$

and proves that (1) has a solution which satisfies a similar functional equation of lower order if and only if one of the ratios a_i/b_j is of the form q^k where k is a non-negative integer. If $j=n$, the reduced functional equation has a right hand side, otherwise the right hand side is zero. If one of the ratios $a_i/b_j=q^k$ where k is a negative integer, then (1)

has a solution which is a polynomial in x multiplied by a power of x. *A. Erdélyi* (Pasadena, Calif.).

Kulik, S. Linear difference equations with boundary conditions. Proc. London Math. Soc. (2) **53**, 331–344 (1951).

This paper studies the linear difference equation

$$(1)\quad L(y)=p_0(x)y(x+k)+p_1(x)y(x+k-1)+\cdots+p_k(x)y(x)=U(x),$$

$x=0, 1, \cdots, n-k$, where the coefficients and the right-hand member are defined and $p_0(x)\neq 0$ for the given values of x. It is required to find a solution of the equation which satisfies also the linear homogeneous boundary conditions

$$(2)\qquad \sum_{m=0}^{n}\alpha_{jm}y_m=0,\quad j=1, 2, \cdots, k,$$

where at least one coefficient in each equation does not vanish. A Green's function is defined, its properties investigated, and the solution of the boundary value problem expressed in the form of a summation involving the Green's function. The problem is then extended to cases in which the boundary conditions (2) are nonhomogeneous, the equation (1) is rearranged to include a parameter, and finally to cases in which the coefficients of $L(y)$ satisfy certain symmetry conditions. *P. E. Guenther.*

Duff, G. F. D. A development in the theory of the F-equation. Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 146–154. University of Toronto Press, Toronto, 1951. $6.00.

In discussing the functional equation

$$\partial F(z, \alpha)/\partial z=F(z, \alpha+1),$$

Truesdell [Unified Theory of Special Functions . . . , Princeton Univ. Press, 1948; these Rev. 9, 431] showed that a function $\phi(\alpha)$ which is regular in a right half-plane and is subjected to a growth restriction, determines uniquely a solution of the F-equation which reduces to $\phi(\alpha)$ when $z\to z_0$. The author gives a new proof of Truesdell's theorem and investigates the situation when $\phi(\alpha)$ is regular in a left half-plane. The "power function" $\Gamma(\alpha)(z_0-z)^{-\alpha}$, $\Re(\alpha)<0$ shows that there is no uniqueness in this case, but the author succeeds in proving that for a certain class of functions $\phi(\alpha)$, there is a power series solution, and an exponential series solution, of the F-equation, and the two differ by a finite sum of "power functions". Examples involving Riemann's zeta function illustrate the general result. *A. Erdélyi* (Pasadena, Calif.).

Myškis, A. D. On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument. Mat. Sbornik N.S. **28(70)**, 641–658 (1951). (Russian)

The author studies the equation

$$(1)\quad y'(x)+M(x)y(x-\Delta(x))=0\qquad [M(x)\geqq 0,\ \Delta(x)\geqq 0,\ A\leqq x<B,\ -\infty<A<B\leqq\infty].$$

A set E_0 is defined as the set of values $x-\Delta(x)$ not exceeding A, with the point A adjoined. On E_0 there is assigned an initial function $\varphi(x)$; $y(x)$ is to satisfy (1), while one sets $y(x-\Delta(x))\equiv\varphi(x-\Delta(x))$ wherever $x-\Delta(x)<A$; $y(A)=\varphi(A)$; $M(x)$, $\Delta(x)$, $\varphi(x)$ are continuous. It is known [Myškis, Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 99–141 (1949); these Rev. 11, 365] that there is a unique solution on $[A, B)$. Among the many results is the following comparison theorem. Suppose that in addition to (1) one has the equation

$$(2)\quad y'(x)+\bar{M}(x)y(x-\bar{\Delta}(x))=0\qquad [A<x<\bar{B},\ -\infty<A<\bar{B}\leqq\infty]$$

with an initial condition $\bar{\varphi}(x)$ and solution $\bar{y}(x)$ (the conditions on the functions in (2) being of the same kind as in (1)); let $y'(x)<0$ on $[A, C)$ $(A<C<B)$, where $y(x)$ is the solution of (1), while $\bar{y}(A)=y(A)$, $\bar{y}(x)\leqq y(A)$ $(A\leqq x\leqq\bar{C}, A<\bar{C}<\bar{B})$, $\bar{y}(\bar{C})=y(C)$ and $\bar{\Delta}(\bar{x})\leqq\Delta(x)$, $\bar{M}(\bar{x})\leqq M(x)$ (for $A\leqq\bar{x}\leqq\bar{C}$, $A\leqq x\leqq C$, $x\leqq\bar{x}$), $\bar{\varphi}(\bar{x})\leqq\varphi(x)$ ($\bar{x}$ in $\bar{E}_0$, x in E_0, $x\leqq\bar{x}$); let χ be defined on the set W of those x in $[A, \bar{C}]$ for which $\bar{y}(x)\geqq y(C)$, so that $A\leqq\chi(x)\leqq C$, $y(\chi(x))=\bar{y}(x)$; under the above conditions one concludes as follows: (a) If $\bar{y}(x)<y(C)$ for x in $[A, \bar{C}]$, then $\chi'(x)$ exists and $\chi'(x)\leqq 1$; (b) $C\leqq\bar{C}$; (c) $\bar{y}'(x)\geqq y'(\chi(x))$ in W; (d) $y(x)\leqq\bar{y}(x)$ $(A\leqq x\leqq C)$. The author also studies the behavior of solutions on an interval on which the sign is preserved, occurrence of oscillations when "retardations" are large, the behavior of solutions on an interval of their oscillation, damping of the solutions when the retardations are small. *W. J. Trjitzinsky.*

Aczél, J. Einige aus Funktionalgleichungen zweier Veränderlichen ableitbare Differentialgleichungen. Acta Univ. Szeged. Sect. Sci. Math. **13**, 179–189 (1950).

It is shown that, under suitable differentiability assumptions, the solutions of certain functional equations in several variables can be reduced to solutions of differential equations. Thus, while the general solutions of the difference equations are not obtained, the smooth solutions are readily found and are given explicit integral formulation. The difference equations treated are

$$(*)\qquad \begin{aligned} z[z(x, y), z(u, v)]&=z[z(x, u), z(y, v)],\\ z[z(x, y), t]&=z[x, z(y, t)],\\ z[z(x, y), t]&=z(x, y+t).\end{aligned}$$

It is shown, for example, that for functions of class C'' the equation (*) is equivalent to

$$\frac{z_{xy}}{z_y}-\frac{z_{xx}}{z_x}=X(x),\quad \frac{z_{xy}}{z_x}-\frac{z_{yy}}{z_y}=X(y),\quad \frac{z_{xy}}{z_xz_y}=X(z),$$

and accordingly that the most general strictly monotone function of class C'' which is a solution of (*) is of the form $z=F^{-1}[aF(x)+bF(y)+c]$, where

$$F(t)=\int^t e^{-\int X(z)\,dz}.$$

E. F. Beckenbach (Princeton, N. J.).

Aczél, J., Kalmár, L., et Mikusiński, J. G.-. Sur l'équation de translation. Studia Math. **12**, 112–116 (1951).

The functional equation in question is

$$(1)\qquad f[f(x, u), v]=f(x, u+v),$$

and its solutions are characterized under various assumptions regarding f. It is supposed that the function $f(x, u)$ is defined for $a<x<b$, $-\infty<u<+\infty$ (with $a=-\infty$, $b=+\infty$ permitted) and that its values lie in (a, b). A typical result: Let $f(x, u)$ satisfy the following four conditions: (α) For some number $x_0\in(a, b)$, f satisfies (1) for $x=x_0$ and $-\infty<u, v<+\infty$; (β) $\Omega(u)=f(x_0, u)$ is strictly monotonic; (γ) $\lim_{u\to-\infty}\Omega(u)=a$, $\lim_{u\to+\infty}\Omega(u)=b$ or $\lim_{u\to-\infty}\Omega(u)=b$, $\lim_{u\to+\infty}\Omega(u)=a$; ($\delta$) $f(x, u)$ is strictly monotonic in x for a non-denumerable set of values of u. Then $f(x, u)$ has the form (2) $f(x, u)=\Omega[\omega(x)+u]$, where $\omega(x)$ is continuous and

strictly monotonic and carries (a, b) into $(-\infty, +\infty)$. The proof is made by showing that Ω is continuous and satisfies (3) $f[\Omega(u), v]=\Omega(u+v)$, so that (2) follows on taking $u=\omega(x)$ as the inverse of $x=\Omega(u)$. Other results lead to the same solution (2) on the assumption of weaker hypotheses.

I. M. Sheffer (State College, Pa.).

Integral Equations

*Vekua, N. P. **Sistemy singulyarnyh integral'nyh uravnenii i nekotorye graničnye zadači. [Systems of Singular Integral Equations and Some Boundary Problems].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 252 pp.

In this book the author studies Hilbert boundary value problems and systems of singular integral equations in the complex plane, the kernels being of Cauchy type and the integrations in the sense of principal values. Chapter I reproduces essentially the author's joint paper with N. I. Mushelišvili [Trav. Inst. Math. Tbilissi [Trudy Tbilissi Mat. Inst.] **12**, 1–46 (1943); these Rev. **6**, 272]; this refers to Hilbert boundary value problems, singular integral systems and theorems of Noether type, the problems of regularisation and equivalence, and the notion of the index. Chapter II presents the solutions of the Hilbert boundary value problems for several unknown functions in the case of discontinuous coefficients; this is applied to a corresponding system of singular integral equations with discontinuous coefficients; in this connection, use is made of the notion of the index, canonic systems of solutions, of a suitable classification of the solutions, of characteristic systems of singular integral equations. Typical are theorems asserting that, under suitable conditions, a singular integral equation is equivalent to an essentially regular Fredholm integral equation; there are theorems of Noether type. The contours, finite in number, are "smooth", closed or open, without common points. The hypotheses are that the functions are of Hölder type except at a finite number of discontinuities of the first kind. Chapter III contains applications of the preceding development to the Riemann-Hilbert problem for systems of analytic functions, the Riemann problem in the case of discontinuous coefficients and to the boundary value problems for elliptic differential equations. Chapter IV involves some generalizations for the Hilbert boundary value problems and for systems of singular integral equations; this includes consideration of "corner" points on the contours involved.

W. J. Trjitsinsky (Paris).

Hadamard, Jacques. Les fonctions de classe supérieure dans l'équation de Volterra. J. Analyse Math. **1**, 1–10 (1950). (French. Hebrew summary)

The paper deals with "functions of class α", i.e. a function $f(x)$ whose derivatives f^p satisfy an inequality $|f^p|<kR^{-p}\Gamma(\alpha p)$ for every positive integer p and suitable values k, R. It is seen that the class of functions remain invariant under fundamental functional operations; e.g., if in a Volterra integral equation the kernel and the given function are of one class, then the solution and the resolving kernel are of the same class. New light is thrown on previous investigations by Holmgren, Gevrey and others, particularly in connection with the rôle played by functions of class 2 in the theory of the equation of heat conduction.

The results can be used to give a proof of a theorem by Gevrey, namely that a solution of the heat equation can be extended across a non-characteristic line L of class 2 if its value on this line is a function of class 2. For one can, following Holmgren, write a Volterra integral equation of class 2 for the normal derivative of the solution on L; so by the above result the normal derivative itself is of class 2. The Cauchy problem can be solved for the heat equation if the non-characteristic initial line and the Cauchy data are of class 2.

R. Courant (New York, N. Y.).

Landweber, L. An iteration formula for Fredholm integral equations of the first kind. Amer. J. Math. **73**, 615–624 (1951).

The author considers $\int_a^b k(x, y)g(y)dy=f(x)$ and the related equation $Kg=F$, where $K=k^*k$ and $F=k^*f$ in operator notation, and k, f are continuous. The iteration scheme $g_n=g_{n-1}+F-Kg_{n-1}$, with g_0 piecewise continuous, is shown to lead to an approximate solution, in the sense that (i) $Kg_n\to F$ uniformly, (ii) $kg_n\to f$ in mean square if f can be expanded in Schmidt functions ϕ_i (characteristic functions of kk^*), (iii) $kg_n\to f$ uniformly if $kg=f$ has a piecewise continuous solution g. Under further restrictions, it is shown that g_n tends to a solution g of $kg=f$.

G. E. H. Reuter (Manchester).

Goldfain, I. A. Sur une classe d'équations intégrales linéaires. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 104–112 (1946). (Russian. French summary)

If $K(x, y)$ is a real L^2 kernel, with characteristic values $\lambda_1, \lambda_2, \cdots$, it is well known [I. Schur, Math. Ann. **66**, 488–510 (1909)] that

$$(1)\qquad \sum_{n=1}^{\infty}|\lambda_n|^{-2}\leqq\int\int[K(x, y)]^2dxdy,$$

and it is also known (the author attributes the result to Hellinger) that equality holds in (1) if $K(x, y)$ is a normal kernel, i.e.

$$\int K(x, t)K(y, t)dt=\int K(t, x)K(t, y)dt$$

for almost all (x, y). The author proves the converse result, that equality in (1) implies that $K(x, y)$ is normal. The minor error between equations (7) and (8) does not affect the validity of the author's proof.

F. Smithies.

Tsuji, Masatsugu. On the compactness of space L^p $(p>0)$ and its application to integral equations. Kōdai Math. Sem. Rep. **1951**, 33–36 (1951).

The author extends M. Riesz's necessary and sufficient conditions for compactness of a subset of L^p $(p\geqq 1)$ to the case $0<p<1$. This is then applied to prove the nondeterminantal Fredholm theorems for the integral equation $\varphi(x)-\lambda\int_a^b K(x, y)\varphi(y)dy=f(x)$ where K is square-integrable in $a\leqq x\leqq b$, $a\leqq y\leqq b$, $f\in L^2$ and the equality holds almost everywhere.

J. V. Wehausen (Providence, R. I.).

Lehmann, N. Joachim. Der Zusammenhang allgemeiner Randwertaufgaben mit der Integralgleichungstheorie. Math. Nachr. **5**, 139–160 (1951).

This paper is concerned primarily with the Schmidt theory of linear integro-differential equations of the form

$$y(s)=\lambda\sum_{\nu,\rho=0}^{n}\left\{\int_a^b K^{\nu,\rho}(x, s)y^{(\rho)}(s)d\sigma_{\nu\rho}(s)+\sum_{i,k}^{a_\nu}a_{\nu i}^{k}K^{\nu,\rho}(x, x_i)y^{(\rho)}(x_i)\right\},$$

where: (i) $K(x, s)$ is real-valued, symmetric, and such that $K^{\nu,\rho}(x, s) = \partial^{\nu+\rho}K(x, s)/(\partial x^{\nu}\partial s^{\rho})$, $\nu, \rho = 0, 1, \cdots, n$, are continuous in (x, s) on $a \leqq x, s \leqq b$; (ii) $\{x_i\}$ is a finite sequence of points on $a \leqq x \leqq b$; (iii) $a_{\nu\rho}^{ik}$ are real constants, and $\sigma_{\nu\rho}(s)$ are real-valued functions of bounded variation on $a \leqq s \leqq b$ such that on a suitable linear function space $\mathfrak{D}$ the inner product

$$(u, v) = \sum_{\nu,\rho=0}^{n} \left\{ \int_a^b u^{(\nu)}(s)v^{(\rho)}(s)d\sigma_{\nu\rho}(s) + \sum_{i,k} a_{\nu\rho}^{ik} u^{(\nu)}(x_i)v^{(\rho)}(x_k) \right\}$$

is symmetric and positive definite. As indicated in the introduction, the considered type of equation includes as a very special case an integro-differential equation that is equivalent to a definite self-adjoint boundary problem of the form treated by Kamke [see, in particular, Math. Z. **48**, 67–100 (1942); these Rev. **4**, 277]. The latter portion of the paper is concerned with the case where (u, v) is merely positive semi-definite on $\mathfrak{D}$. It is to be remarked that many of the results obtained are immediate from the theory of completely continuous symmetric transformations in Hilbert space, although the author makes no references to papers on this subject. *W. T. Reid* (Albuquerque, N. M.).

Platone, Giulio. Teorema d'unicità per le equazioni integrali non lineari ottenute da funzioni di composizioni a nucleo sommabile. Pont. Acad. Sci. Acta **14**, 1–8 (1950). (Italian. Latin summary).

Let $f \overset{A}{*} g$ denote the convolution (with weight A)

$$\int_E A(\xi)f(x, \xi)g(\xi, y)d\xi,$$

$\overset{A}{f^r}$ the $(r-1)$ times iterated convolution of f by itself, $[I]_A$ the class of essentially bounded functions $f(x, y)$ for which the convolution with weight A is commutative, and $P_r(x, y, \lambda)$ the convolution polynomial

$$\sum a_{i_1 \cdots i_n}^{(r)} \lambda_1^{i_1} \cdots \lambda_n^{i_n} \overset{A}{k_1}^{i_1} \cdots \overset{A}{k_n}^{i_n}$$

of the kernels $k_i(x, y) \in [I]_A$ $(0 \leqq i_1 + \cdots + i_n \leqq s_r)$. It is then proved that the equation

$$a_1 Y(x, y, \lambda) + \sum_{r=1}^{m} \int_E A(\xi) P_r(x, \xi, \lambda) \overset{A}{Y^r}(\xi, y, \lambda)d\xi - k_0(x, y) = 0$$

with $a_1 \neq 0$ has, for sufficiently small values of the λ_i, one and only one solution $Y \in [I]_A$. To prove the existence of a solution it is essential that

$$(\partial/\partial z)[a_1 z + \sum_r (\sum a_{i_1 \cdots i_n}^{(r)} z_1^{i_1} \cdots z_n^{i_n}) z^r] \neq 0$$

for $z = z_1 = \cdots = z_n = 0$. It seems to the reviewer that this is not necessarily satisfied under the hypotheses of the theorem. *M. Golomb* (Lafayette, Ind.).

Voskresenskiĭ, E. P., and Sobolev, V. I. On a class of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) **79**, 717–718 (1951). (Russian)

A result of Valnberg [same Doklady **75**, 609–612 (1950); these Rev. **12**, 713] on the existence of eigenvalues μ and eigenfunctions $x(t)$ for the equation

$$\mu x(t) = \int_E K(t, s)g(s, x(s))ds$$

is extended to the case where $K(t, s)$ instead of being symmetric is of the form $a(t)b(s)Q(t, s)$ with $Q(t, s)$ symmetric and $a(t)$, $b(s)$ positive. *M. Golomb* (Lafayette, Ind.).

Valnberg, M. M. The existence of characteristic functions for nonlinear integral equations with nonpositive kernels. Doklady Akad. Nauk SSSR (N.S.) **78**, 1077–1080 (1951). (Russian)

With the aid of approximating finite-dimensional minimum problems the author proves the following theorem. If the real symmetric kernel $K(x, y) \in L_2(B \times B)$ has a finite number of positive eigenvalues, and if the operator $hu = g(u(x), x)$ is continuous on $L_2(B)$ while

$$\operatorname{sign} g(u, x) = \operatorname{sign} u, \quad |g(u, x)| \geqq \alpha|u|, \quad \alpha > 0,$$

then the equation

$$\mu u(x) = \int_B K(x, y)g(u(y), y)dy$$

has at least a denumerable number of eigenfunctions $u_1(x), u_2(x), \cdots$ belonging to positive eigenvalues $\mu_1, \mu_2, \cdots$, where $\|u_1\| < \|u_2\| < \cdots$. *M. Golomb* (Lafayette, Ind.).

√* **Davison, B., Kushneriuk, S. A., and Seidel, W. P. Influence of a small black cylinder upon the neutron density in an infinite non-capturing medium.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 172–195. University of Toronto Press, Toronto, 1951. $6.00.

The contents of this paper are based on an earlier report [National Research Council of Canada. Atomic Energy Project. Division of Research. **MT-207** (N.R.C. no. 1868) (1948); these Rev. **11**, 599].

Kuznecov, E. S. Radiative equilibrium of a gaseous sphere surrounding an absolutely black body. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 3, 69–93 (1951). (Russian)

The author considers the equation of radiative transfer, in plane polar coordinates, and derives from it certain integral equations governing the distribution of total radiation (i.e., of the temperature) within a gaseous shell surrounding a sphere which radiates like a black body. In part I of his investigation the author formulates the basic equations of his problem; part II is devoted to an investigation of their specific form in the case of a constant absorption coefficient throughout the gaseous shell, while in part III this coefficient is allowed to become an arbitrary function of radial distance. In the concluding part IV, the author considers the limiting form of his equations if the curvature of layers of constant absorption is allowed to vanish. *Z. Kopal* (Manchester).

Kuznecov, E. S. A general method for the construction of approximate equations of transfer of radiative energy. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 4, 71–91 (1951). (Russian)

The author's aim in this paper is to construct an approximate solution of the equation of transfer of radiation, in an atmosphere stratified in plane-parallel layers, incident from outside and scattered in accordance with a general indicatrix which is assumed to be expansible in terms of Legendre polynomials of the angle between incident and scattered beams, with arbitrary coefficients. The author seeks his solution by assuming an expansion in terms of zonal harmonics for the emergent intensity $I(\tau, \theta)$ as well (following an earlier work by Gratton [Mem. Soc. Astr. Ital. (3) **10**, 309–337 (1937)] and Chandrasekhar [Astrophys. J. **99**, 180–190 (1944); these Rev. **6**, 76], who both employed essentially the same analytic means for the construction of an approximate solution for radiative transfer of reemitted light), and considers in more detail the case of a linear indicatrix (a case which was

previously treated by Ambarzumian [Acad. Sci. USSR. J. Phys. 8, 65–75 (1944); these Rev. 7, 97]. *Z. Kopal.*

Chandrasekhar, S., and Münch, G. The theory of the fluctuations in brightness of the Milky Way. III. Astrophys. J. 114, 110–122 (1951).

In this paper an integral equation (derived previously by the authors [same J. 112, 380–392 (1950); these Rev. 12, 644]) governing the fluctuations in brightness of the Milky Way is explicitly solved for the case in which the stellar system under consideration extends to a finite distance in the direction of the line of sight, and when all absorbing clouds are equally transparent. The derived theoretical distributions are illustrated on several diagrams.

Z. Kopal (Manchester).

Vlasov, A. A., and Bazarov, I. P. On the theory of strata. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 1098–1108 (1950). (Russian)

Considerations of the passage of electricity through gases, particularly the occurrence of space-periodic phenomena, lead the authors to set up a complicated non-linear integro-differential equation involving retarded values. A heuristic process of solution in terms of a Fourier series leads, after calculations involving principal-value integrals, to periods showing some degree of agreement with experiment. The emphasis of the paper is primarily physical.

F. V. Atkinson (Ibadan).

Functional Analysis, Ergodic Theory

Kelley, J. L. Note on a theorem of Krein and Milman. J. Osaka Inst. Sci. Tech. Part I. 3, 1–2 (1951).

By using Zorn's lemma rather than transfinite induction, the author gives a simple proof of the following extension of the Krein-Milman theorem [Studia Math. 9, 133–138 (1940); these Rev. 3, 90]: Let L be a real linear topological space such that: (A) For each compact convex C and each x not in C there is a continuous linear functional whose value at x is greater than its value at any point of C. Then (T) every compact convex $K \subset L$ is the closed convex hull of its set of extreme points. Calling "supporting set" every compact convex $S \subset K$ such that if an interior point of a segment in K is in S, then so is the whole segment, the author observes that every intersection of supporting sets is a supporting set, and hence by Zorn's lemma K contains a minimal supporting set. By use of (A) this is shown to be one-pointed, and hence (T′) every compact convex set in L has at least one extreme point. The proof is completed by (T′) in conjunction with (A). Similar proofs have recently been given by Artin [Piccayune Sentinel, Bloomington, Indiana, Dec. 16, 1950] and Hotta [see the following review]. [Reviewer's note: The author remarks that "it is rather the simplicity of the proof than the slight generality which seems interesting". Nevertheless, it may be observed that to prove (T′) it suffices for L to be a linear space with associated topology such that for each pair x, y of distinct points of L there is a continuous linear f for which $f(x) \neq f(y)$. To prove (T) the following additional hypothesis is sufficient: for each $y \in L$ and $r \in R$ (reals), $y+rx \mid x \in L$ and $ry \mid r \in R$ are continuous.] *V. L. Klee, Jr.* (Princeton, N. J.).

Hotta, Jyôji. A remark on regularly convex sets. Kôdai Math. Sem. Rep. 1951, 37–40 (1951).

Let E be a normed linear space and C a subset of its conjugate space E^*. The author proves (1) C is regularly convex if and only if it is convex and *-weakly closed; (2) if C is *-weakly compact, it has an extreme point. He mentions that for separable E, (1) is due to Krein and Šmulian [Ann. of Math. (2) 41, 556–583 (1940); these Rev. 1, 335]. (For general E, (1) was proved by Šmulian [Rec. Math. [Mat. Sbornik] N.S. 7(49), 425–448 (1940); these Rev. 2, 102], but the author's proof is perhaps more direct.) His proof of (2) uses Zorn's lemma rather than the transfinite induction of the original Krein-Milman proof [Studia Math. 9, 133–138 (1940); these Rev. 3, 90] and is correspondingly simpler. Similar proofs have been given (slightly before the author's) by Kelley [see the preceding review] and Artin [Picayune Sentinel, Bloomington, Indiana, Dec. 16, 1950].

V. L. Klee, Jr. (Princeton, N. J.).

***da Silva Dias, Candido. Espacos vectoriais topológicos e sua applicação na teoria dos espaços funcionais analíticos. [Topological Vector Spaces and Their Application in the Theory of Analytic Functional Spaces].** Thesis, University of São Paulo, 1951. ii+66 pp.

The first part of this thesis is a summary (with some proofs) of known results from the theory of duality in topological vector spaces. In the second part the author applies that theory to the Fantappié theory of "analytical functionals", going beyond the recent progress made in that field by J. Sebastião Silva [Portugaliae Math. 9, 1–130 (1950); these Rev. 11, 524]. On the Riemann sphere, let F be a closed subset (not identical to the whole sphere), and let (F) be the set of all functions which are defined and analytic in some open set containing F (and depending on the function) with the additional restriction that such a function is 0 at the point at infinity if that point is in F. Two functions in (F) are equivalent if they coincide on some neighborhood of F, hence a definition of the set $[F]$ of equivalence classes, which is a vector space in an obvious way. Let (A_n) be a fundamental system of open neighborhoods of F such that $\bar{A}_{n+1} \subset A_n$; and let $[F_n]$ be the subset of $[F]$ consisting of classes which have a representative which is defined and bounded in $\bar{A}_n$; $[F_n]$ is a Banach space for the topology of uniform convergence in A_n. Now $[F]$ is the union of the spaces $[F_n]$, and the author takes as a topology on $[F]$ the "inductive limit" of the topologies of the $[F_n]$ [G. Köthe, Math. Z. 52, 627–630 (1950); these Rev. 12, 417]. He then proves that the dual of that space can be identified with the space $[O]$ of all functions defined and analytic in the open complement O of F (and 0 at the point at infinity if it is in O); the duality being expressed by the Fantappié formula $\langle f, u \rangle = (1/2\pi i) \int_\Gamma f(z)u(z)dz$ for $f \in (F)$ and $u \in [O]$, Γ being a Jordan curve (or system of a finite number of Jordan curves) containing F in its interior. The strong topology on $[O]$ (as the dual of $[F]$) is shown to be the topology of uniform convergence on compact subsets of O; $[O]$ and $[F]$ are Montel spaces ($[O]$ being in addition metrizable), and $[F]$ is the dual of $[O]$. In an appendix the author shows how Runge's theorem on approximation by rational functions or polynomials is an immediate consequence of his theory.

J. Dieudonné (Nancy).

Levi, Friedrich Wilhelm. Normierte Moduln. Math. Nachr. 5, 379–392 (1951).

The author studies functionals A on a real linear space M (not necessarily of finite dimension) of the form

$A(x)=\sum_{i=1}^{m}N(r_i(x))-\sum_{j=1}^{n}N(s_j(x))$, where the r_i and s_j are real linear functions on M and N is a norm on M (called "proper" when $N(a)=0$ implies $a=0$). A typical question is whether the positive-definiteness or semidefiniteness of A for one norm implies the corresponding property for some other norm. The set of all semidefinite A (i.e. $A(x)\geqq 0$, $x\in M$) form a semimodule $S(N, M)$ over the positive reals, and the set of all such semimodules constitutes a complete lattice. If M has finite dimension >2, then the cardinality of $S(N, M)$ is equal to that of the continuum. Two norms N and N' on M are called equivalent if $S(N, M)=S(N', M)$, and the equivalence of norms is studied, as well as bundles of linearly independent norms in this connection. If N_p is the Minkowski norm of index p relative to a fixed choice of basis ($1\leqq p\leqq\infty$; N_∞ gives the sup), then: a) if S is positive definite for N_∞, it is positive definite for every proper norm; b) $S(N_1, M)=S(N_2, M)$. Certain numerical invariants of the class $S(N, M)$ are given. *I. E. Segal.*

Klee, V. L., Jr. Some characterizations of reflexivity. Revista Ci., Lima **52**, nos. 3–4, 15–23 (1950).

The author proves several facts concerning reflexive spaces. Some of them are given below: 1) A space E is reflexive if and only if there is a hyperplane H in E such that if B is a bounded closed centered convex body in E at zero distance from H, then B intersects H. Thus 2) E is reflexive if and only if in each of its isomorphs every continuous linear functional attains its maximum on the unit sphere. 3) The following are equivalent: (i) E is reflexive; (ii) if X and Y are bounded closed convex subsets of E, then $X+Y$ is closed; (iii) if X and Y are bounded closed convex subsets of E, then the convex hull of $X\cup Y$ is closed. 4) The space (m) has an isomorph which is not equivalent to any adjoint space. The author has submitted to the editors a list of misprints in his paper. The more serious are: On p. 17, l. 8 read $(1-2^{-|n|})x$; on p. 17, l. 1*, read $t^i_j(1-2^{-j})$; p. 20, footnote, the first $\overline{\text{conv}}\ U$ should read $\overline{\text{conv}}\ U$; p. 22, l. 1, (L) should be (l). *E. R. Lorch* (New York, N. Y.).

Alexiewicz, A., and Orlicz, W. On analytic vector-valued functions of a real variable. Studia Math. **12**, 108–111 (1951).

Let $x(t)$ be defined from a closed real interval $[a, b]$ to a Banach space X. Then $\Gamma\subset X^*$ is said to be fundamental if there exist positive constants α, k such that $\sup_{x^*\in\Gamma, \|x^*\|\leq k}|x^*x|\geqq\alpha\|x\|$ for all $x\in X$. The set Γ is strictly fundamental if for a sequence $\{x_n\}\subset X$ $\limsup_n|x^*x|<\infty$ for $x^*\in\Gamma$ implies $\limsup_n\|x_n\|<\infty$. It is shown that if $d^2(x^*x(t))/dt^2$ exists and is bounded in each closed subinterval of $[a, b]$, $x(t)$ possesses a derivative in $[a, b]$. It is also proved that analyticity of $x^*x(t)$ on $[a, b]$ for $x^*\in\Gamma$ implies analyticity of $x(t)$ on $[a, b]$. *R. E. Fullerton.*

Alexiewicz, A., and Orlicz, W. Remarks on Riemann-integration of vector-valued functions. Studia Math. **12**, 125–132 (1951).

Let $x(t)$ be defined as in the preceding review. The Riemann-Graves (RG) integral of $x(t)$ over $[a, b]$ is defined as the limit of sums $S(\pi)=\sum x(\tau_i)|\delta_i|$, $\tau_i\in\delta_i$ over a normal sequence of partitions $\pi=\{\delta_1, \delta_2, \cdots, \delta_n\}$ of $[a, b]$. Various theorems are proved, chiefly by example, to illustrate the difference between the cases of a general Banach space X and the space of the reals as a range space for $x(t)$ with regard to integrability. (RG)-integrable functions separably valued but everywhere discontinuous in $[a, b]$ and separably valued but everywhere weakly discontinuous in $[a, b]$ are constructed and it is shown that neither weak integrability nor weak continuity of $x(t)$ implies (RG)-integrability of $x(t)$. *R. E. Fullerton* (Madison, Wis.).

Alexiewicz, A. Continuity of vector-valued functions of bounded variation. Studia Math. **12**, 133–142 (1951).

The author proves various theorems about the set of points of discontinuity and weak discontinuity of vector valued functions of bounded variation defined over a real interval $[a, b]$. If $x(t)$ is separably valued and BV it may be everywhere discontinuous but weakly discontinuous on at most a countable set of points. Other theorems of this type show that if $x(t)$ is BV and compactly valued its set of points of discontinuity is at most countable and if $x(t)$ is BV with range in a fundamentally separable weakly complete space, the set of points of weak discontinuity is at most countable. *R. E. Fullerton* (Madison, Wis.).

Alexiewicz, A. On sequences of operations. III. Studia Math. **12**, 84–92 (1951).

[Parts I and II appeared in Studia Math. **11**, 1–30, 200–236 (1950), respectively; these Rev. **12**, 418, 507.] In this paper the author continues the study of A-spaces started in Part II with a study of polynomial operations (still supposed continuous). He shows that the conditions, shown in Part II to be sufficient for the statements I^l, II^l, III_1^l and III_2^l to hold, are also sufficient for the statements I^m, II^m, III_1^m and III_2^m respectively to hold, where the latter statements are obtained from the former by replacing "linear" by "polynomial of degree at most m" in I and II and by "polynomial of degree at most m_p" in III. *A. F. Ruston.*

Alexiewicz, A. On sequences of operations. IV. Studia Math. **12**, 93–101 (1951).

[See the preceding review for part III and references to parts I and II.] In this paper the author considers linear and polynomial operations from a A-space into the space S of measurable functions over a set T of finite measure, equivalent functions being identified. With the norm

$$\|x\|=\int_T\frac{|x(t)|}{1+|x(t)|}dt,$$

this is an F-space. Convergence in this norm is identical with asymptotic convergence. Convergence almost everywhere in T [misprinted S] is also considered. Theorems of Banach and Saks on such questions as the condensation of singularities are generalised. *A. F. Ruston* (London).

* **Dunford, Nelson. Spectral theory.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 203–208. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Let T be a bounded linear operator on a Banach space R with values in R, and let $T_\lambda=(\lambda I-T)^{-1}$ be its resolvent. Let C be an oriented rectifiable boundary of a finite number of connected open sets containing the spectrum $\sigma(T)$ of T. If f is regular on and within C, the mapping $f\to f(T)=(2\pi i)^{-1}\int_C f(\lambda)T_\lambda d\lambda$ is a ring-homomorphism such that $f(T)=T^n$ for $f(\lambda)=\lambda^n$ [see N. Dunford, Bull. Amer. Math. Soc. **49**, 637–651 (1943); these Rev. **5**, 39]. Suggested by the elementary divisor theory, the author considers the possibility of the representation

$$f(T)=\sum_{n=0}^{\infty}\int_{\sigma(T)}(n!)^{-1}f^{(n)}(\lambda)(T-\lambda I)^n dE_\lambda,$$

with a Boolean algebra of projections E_τ ($E_\tau T = TE_\tau$) which is a homomorphic map of a Boolean algebra of Borel sets τ of $\sigma(T)$. The author calls T a spectral operator if E_τ is countably additive in the weak operator topology. Many interesting conditions for T to be such an operator are given without proofs. For example, in a reflexive space R, T is a spectral operator such that $f(T)=\sum_{n=0}^{m-1}\int_{\sigma(T)}(n!)^{-1}f^{(n)}(\lambda)(T-\lambda I)^n dE_\lambda$ if and only if the following two conditions are satisfied:

$$\text{(i)} \qquad \left|(2\pi)^{-1}\int_C x^*(T_\lambda x)d\lambda\right| \leqq K\|x\|\cdot\|x^*\|$$

for $x^* \varepsilon R^*$, (ii) $d(\lambda)^m\|T_\lambda\| \leqq K$ where $d(\lambda)$ is the distance from λ to $\sigma(T)$. Self-adjoint or normal operators T in a Hilbert space are the examples for the special case $m=1$.

K. Yosida (Nagoya).

*Taylor, Angus E. **Spectral theory of unbounded closed operators.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 267–275. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Similarly as in N. Dunford's paper [see the preceding review], the author starts with the ring homomorphism $f(\lambda)\to f(T)=(2\pi i)^{-1}\int f(\lambda)T_\lambda d\lambda$, $T_\lambda=(\lambda I-T)^{-1}$ = the resolvent of T. In order to include unbounded operators T, the author replaces the definition of $f(T)$ above as follows. The extended spectrum $\sigma_e(T)$ of T is the ordinary spectrum $\sigma(T)$, together with the point $\lambda=\infty$ if and only if T is unbounded. Assuming $f(\lambda)$ to be regular on a domain $\Delta(f)$ which contains $\sigma_e(T)$, the extended $f(T)$ may be defined by $f(T)=f(\infty)I+(2\pi i)^{-1}\int_{B(D)}f(\lambda)T_\lambda d\lambda$, where the oriented boundary $B(D)$ surrounds a suitable domain D which contains $\sigma_e(T)$, such that the closure D^a lies in $\Delta(f)$. It is proved that the new $f(T)$ coincides with the old one when T is bounded by defining f arbitrarily (but so as to be regular) in a restricted neighborhood of $\lambda=\infty$, and that a satisfactory operational calculus may be founded upon this new $f(T)$ under the condition that $\sigma_e(T)$ does not cover the whole complex plane. For example, if f has a zero of finite order at $\lambda=\infty$, but not other zeros in $\sigma_e(T)$, then $f(T)$ defines a one-to-one mapping of the entire Banach space R onto $D_e(T)=\{x \varepsilon R;\ x, Tx, \cdots, T^{n-1}x$ all in the domain of $T\}$.

K. Yosida (Nagoya).

Zwirner, Giuseppe. Elementi uniti di trasformazioni funzionali e teoremi di differenziabilità rispetto a un parametro. Ann. Triestini. Sez. 2 (4) 3(19) (1949), 83–104 (1950).

The author considers equations of the form $y=T(y,x)$, where x and y are variables in normed linear spaces, and obtains two theorems on the differentiability of the solution $y=y(x)$. These are closely related to a result of Hildebrandt and Graves [Trans. Amer. Math. Soc. 29, 127–153 (1927)], but the hypotheses and conclusion are weaker than those of the theorems explicitly stated by those authors. An application is made to second order differential equations.

L. M. Graves (Chicago, Ill.).

Kračkovskiĭ, S. N., and Gol'dman, M. A. Null elements and null functionals of completely continuous operators. Latvijas PSR Zinātņu Akad. Vēstis. 1950, no. 6 (35), 87–100 (1950). (Russian. Latvian summary).

This paper extends to linear normed spaces some of the nondeterminental results of Fredholm integral equation theory, following in particular ideas developed by F. Riesz for special spaces [Acta Math. 41, 71–98 (1916)]. The results seem to be well known for the most part [cf. Banach, Théorie des opérations linéaires, Warsaw, 1932, chap. 10; Hildebrandt, Bull. Amer. Math. Soc. 37, 185–212 (1931), especially pp. 196–202]. *J. V. Wehausen.*

Krasnosel'skiĭ, M. A. On the problem of branch points. Doklady Akad. Nauk SSSR (N.S.) 79, 389–392 (1951). (Russian)

Let E be a real Banach space and A an operator on E that leaves the null element θ of E invariant. Then the real number λ is an eigenvalue of A if $f=\lambda Af$ for some eigenvector $f\neq\theta$, and λ is a branch point of A if in every open interval containing λ there are eigenvalues of A to which correspond eigenvectors of arbitrarily small norm. Suppose the linear operator B is the Fréchet differential of A at θ. Then a branch point of A must be an eigenvalue of B. For the converse problem the author has obtained the following results. (1) If A is completely continuous, if $\lambda_{k-1}<\lambda_k<\lambda_{k+1}$ are consecutive eigenvalues of B of odd multiplicity, and if on the boundary of some open set of E containing θ there are no eigenvectors of A for eigenvalues between λ_{k-1} and λ_{k+1}, then at least one of the intervals $\lambda_{k-1}<\lambda<\lambda_k$, $\lambda_k<\lambda<\lambda_{k+1}$ consists entirely of eigenvalues of A. (2) If $L=A+D$ where A is completely continuous and has the Fréchet differential B at θ, and D satisfies the condition $\|Df-Dg\|\leqq q(\rho)\|f-g\|$, where $\rho=\max(\|f\|, \|g\|)$ and $\lim_{\rho\to 0} q(\rho)=0$, then every eigenvalue of B of odd multiplicity is a branch point of L (with a "continuous branch of eigenvectors"). *M. Golomb* (Lafayette, Ind.).

Citlanadze, È. S. On differentiation of functionals. Mat. Sbornik N.S. 29(71), 3–12 (1951). (Russian)

Let E be a reflexive Banach space, with dual E'. The value of $x' \varepsilon E'$ at $x \varepsilon E$ is denoted by (x', x). It is assumed that E and E' have countable bases which are biorthogonal, and some lemmas concerning expansions in terms of these bases are proved. The Fréchet differential of a functional $f(x)$ defined inside the unit sphere of E is defined to be the linear functional $df(x;h)=(Lx,h)$ of h such that $f(x+h)-f(x)=df(x;h)+\omega(x,h)$, where $\omega(x,h)=o(\|h\|)$; it is assumed in the theorems of the paper that $\omega(x,h)=O(\|h\|^2)$. It is shown that if $f(x)$ is weakly continuous, then Lx is weakly continuous and also completely continuous in that it takes the unit sphere into a precompact set. If Lx is weakly continuous, so is $f(x)$. If Lx is strongly continuous, so is $f(x)$, and $f(x)-f(0)=\int_0^1(L(tx),x)dt$.

J. L. B. Cooper (Cardiff).

Michal, Aristotle D. Differential invariants and invariant partial differential equations under continuous transformation groups in normed linear spaces. Proc. Nat. Acad. Sci. U. S. A. 37, 623–627 (1951).

Let $x=(x^1, \cdots, x^m)$ be a point in space of m-dimensions. Likewise let $a=(a^1, \cdots, a^r)$, and let y be a point in a Banach space B. The author considers a continuous transformation group G_r depending on the parameters $a^1, \cdots, a^r$, the transformations being of the form

$$\bar{x}=f(x,a), \quad \bar{y}=\phi(y,a),$$

with f and ϕ admitting total Fréchet differentials up to a certain sufficiently high order. It is assumed that the subgroup $\bar{x}=f(x,a)$ has s ($s<m$) independent numerical scalar invariants, and that G_r has a B-valued scalar invariant $g(y,x)$ whose partial Fréchet differential $g(y,x;\delta y)$ is always solvable for δy. The main concern of the paper is with

differential invariant expressions of the form

$$\Omega(x, y(x), \partial y/\partial x, \cdots, \partial^s y/\partial x^s),$$

where Ω is a function with values in a Banach space B_1, and $y(x)$ is a B-valued function invariant under G_r in the sense that $y(\bar{x})=\phi(y(x), a)$ identically in the x's and a's. If B and B_1 are reflexive, it is shown that Ω is expressible in the form

$$\omega(\eta, \nu(\eta), \partial\nu/\partial\eta, \cdots, \partial^s\nu/\partial\eta^s),$$

where ν is a B-valued function of s numerical variables such that $\nu(\eta(x))=g(y(x), x)$. A converse holds without the reflexivity requirement. The theorem makes it possible to replace certain differential equations $\Omega=0$ by equations $\omega=0$ in a smaller number of independent variables.

A. E. Taylor (Los Angeles, Calif.).

Taylor, A. E. Banach spaces of functions analytic in the unit circle. II. Studia Math. 12, 25–50 (1951).

In the first part of this study [Studia Math. 11, 145–170 (1950); these Rev. 13, 45] author studied complex normed linear spaces of functions analytic in the open unit circle and restricted by a set of axioms. In this part he studies specific instances of these abstract spaces. In the present review we shall use the notations and terminology introduced in the earlier one and in addition define, for a space B of type $\mathfrak{A}_3$, the space B' as the space of all $F\in\mathfrak{A}$ for which $\|F\|'<\infty$. The space B'' is $(B')'$, L^p is $L^p(0, 2\pi)$; $(1/p)+(1/p')=1$; for $p=\infty$ let H^p be the space of those $f\in\mathfrak{A}$ for which $\|f\|_\infty=\sup|f(z)|<\infty$, $z\in\Delta$; for $0<p<\infty$ let H^p be the set of all $f\in\mathfrak{A}$ for which $\|f\|_p=\sup_{0\le r<1}\mathfrak{M}_p[f, r]<\infty$ where $\mathfrak{M}_p=\{(2\pi)^{-1}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\}^{1/p}$. The Cauchy integral of $\varphi\in L^p$ is the integral $f(z)=(2\pi)^{-1}\int_0^{2\pi}(1-ze^{-it})^{-1}\varphi(x)dx$. These spaces H^p, $1\le p\le\infty$, are B-spaces of type $\mathfrak{A}_4$. The Cauchy integral F of a function Φ in L^p is in $(H^p)'$ and $\|F\|\le\|\Phi\|$. If $1\le p<\infty$, then every $F\in(H^p)'$ is the Cauchy integral of a function Φ in $L^{p'}$ with $\|F\|=\|\Phi\|$. The Cauchy integral of a function in L is in $(H^\infty)^0$. If $1<p<\infty$ then the Cauchy integral of a function in L^p is in H^p and the classes $(H^p)'$ and $H^{p'}$ are the same. As Banach spaces they are isomorphic but not equivalent unless $p=2$. Every continuous linear functional γ on H^p is representable in the form

$$\gamma(f)=(2\pi)^{-1}\int_0^{2\pi} f(e^{ix})F(e^{-ix})dx$$

where $F\in H^{p'}$. The element F uniquely determines and is uniquely determined by γ and $\|\gamma\|=\|F\|$. The space of functions $f\in\mathfrak{A}$ which are continuous in the closed unit circle is also discussed. *N. Dunford* (New Haven, Conn.).

Haplanov, M. G. Some properties of an analytic space. Doklady Akad. Nauk SSSR (N.S.) 79, 929–932 (1951). (Russian)

The space is the set A_R ($\bar{A}_R$) of all sequences $x(x_1, x_2, \cdots)$ of complex numbers such that $f(z)=x_1+x_2z+\cdots$ is an analytic function of the complex variable z for $|z|<R$ ($|z|\le R$). In the terminology of Köthe and Toeplitz [J. Reine Angew. Math. 171, 193–226 (1934); or Köthe, Math. Nachr. 4, 70–80 (1951); these Rev. 12, 615] the author's results are: (1) $(A_R)^*=\bar{A}_{1/R}$, $(\bar{A}_R)^*=A_{1/R}$ so that A_R is a perfect coordinate space. (2) $M\subset A_R$ is bounded if and only if the corresponding set of functions has a majorant. (3) $x^n\in A_R$ form a convergent sequence in the Köthe sense if and only if the corresponding functions converge uniformly for $|z|<R$. (4) In A_R, the Köthe limit coincides with the strong Köthe limit. (5) $M\subset A_R$ is bounded if and only if it is strongly bounded. A proof for (2) is sketched.

B. Crabtree (Durham, N. H.).

Tropper, A. Mary. A note on reciprocals of infinite matrices. J. London Math. Soc. 26, 298–301 (1951).

Some remarks are made on matrices termed "Pólya reciprocals" (whose definition here would take too much space), but most of the results concern K_r-, K_c- and H-(Hilbert) matrices. The infinite matrix $A=(a_{ij})$ ($i, j=1, 2, \cdots$) is a K_r-matrix if M exists such that $\sum_{k=1}^n|a_{nk}|<M$ for every n; and if the transpose of A is K_r, then A is a K_c-matrix. Some theorems are proved of which the following are illustrative: (i) If $\{z_k\}$ is a real bounded sequence, and A is real K_r-matrix such that $\lim_{n\to\infty}\sum_{k=1}^n|a_{n,k}-a_{n+1,k}|=0$ and $\lim_{n\to\infty}a_{n,k}=0$, then A carries $\{z_k\}$ into a sequence which either converges or has a continuum of limit points. (ii) If matrix A satisfies the conditions of (i) it cannot have a two-sided K_r-reciprocal. (iii) If A is both an H-matrix and a K_r-matrix, and if $\lim_{n\to\infty}\{\max_{k=1,2,\cdots}|a_{n,k}|\}=0$, then A does not possess a right hand H-reciprocal. Application to the Borel matrix B: $b_{ij}=\varepsilon^{-i}i^j/j!$ shows that it has no left or right Hilbert reciprocal. *I. M. Sheffer* (State College, Pa.).

Sheffer, I. M. On the solution of sum-equations. Amer. J. Math. 72, 835–848 (1950).

The author discusses the system of sum equations (1) $\sum_{j=0}^\infty a_{nj}x_{n+j}=c_n$ ($n=0, 1, \cdots$), where $a_{n0}\neq0$, and for some positive R all the series $A_n(t)=\sum_{j=0}^\infty a_{nj}t^j$ converge for $|t|<R$. The object is to determine solutions of (1), and of the homogeneous system (2) $\sum_{j=0}^\infty a_{nj}x_{n+j}=0$, which are of prescribed "type". (The type $((x_n))$ of $\{x_n\}$ is defined as $\limsup|x_n|^{1/n}$.) Polynomials $H_n(t)$ of degree n are introduced, uniquely determined by an orthogonality condition $(2\pi i)^{-1}\int_C H_n(1/t)t^{k-1}A_k(t)dt=\delta_{nk}$.

For (1), the following are among the results obtained: (a) (Uniqueness) If $\{x_n\}$ is a solution of (1) with $((x_n))<q\le R$, and if the series $\sum_{n=0}^\infty H_n(1/t)u^nA_n(u)$ converges "uniformly" in $|u|<q$, $|t|>r$, (for some $r\ge0$), then x_n is given by $\sum_{n=0}^\infty x_nt^{-n}=\sum_{j=0}^\infty c_jH_j(1/t)$. (b) (Existence) If, for some $r<R$ and some positive q, $((c_n))<\min(q, R)$, and if the series (3) $\Phi(u, 1/t)=\sum_{n=0}^\infty H_n(1/t)u^n$ converges "uniformly" in $|u|<q$, $|t|>r$, then (1) has a solution $\{x_n\}$, with $((x_n))\le r$, given by $\sum_{n=0}^\infty x_nt^{-n}=(2\pi i)^{-1}\int_\gamma(\sum_{j=0}^\infty c_ju^{-j})\Phi(u, 1/t)u^{-1}du$.

In discussing (2) the author uses concepts of ' "linear dependence" ' and 'canonical representation'. The set of functions $\Theta_j(u)$ ($j=1, \cdots, k$) is called "linearly dependent in $|u|<q$" if for some non-trivial set of constants $a_1, \cdots, a_k$ the function $\sum_{j=1}^k a_j\Theta_j(u)$ is analytic in $|u|<p$ for some $p>q$. By a 'canonical representation' of $\Phi(u, 1/t)$ relative to $[q, q_1, q', r]$ (where $0<q<q_1<q'$ and $r\ge0$) is meant a representation (4) $\Phi(u, 1/t)=\sum_{j=1}^k R_j(1/t)\Theta_j(u)+\Phi^*(u, 1/t)$, where $\Phi^*(u, 1/t)=\sum_{n=0}^\infty H_n^*(1/t)u^n$ converges "uniformly" in $|u|<q'$, $|t|>r$, the functions $H_n^*(1/t)$ are analytic in $|t|>r$, the $R_j(1/t)$ are analytic in $|t|>r$ and linearly independent (in the usual sense), the $\Theta_j(u)$ are analytic in $|u|<q$ and "linearly independent" in $|u|<q_1$, and "uniform" convergence of the series in (3) takes place in $|u|<q$, $|t|>r$, but not in $|u|<q_1$, $|t|>r$. It is proved that if there is a canonical representation (4) relative to $[q, q_1, q', r]$ with $r<R$, then for system (2) the set of k sequences $\{x_n^{(j)}\}$ ($j=1, \cdots, k$) defined by $\sum_{n=0}^\infty x_n^{(j)}t^{-n}=R_j(1/t)$ constitutes a set of k linearly independent solutions of type not exceeding r. Under the additional hypothesis that there exist q_2, r_2 with $q_2<q'$ and $r_2>r$ such that the series $\sum_{n=0}^\infty t^nA_n(t)/u^n$ converges

"uniformly" in $|u|>q_2$, $|t|<r_2$ all solutions of (2) having type $\leqq r$ are linearly dependent on these k. Extensions are made to cases where the finite sum in (4) is replaced by an infinite series or an integral. *W. Strodt.*

Graves, Ross E. Additive functionals on a space of continuous functions. II. Ann. of Math. (2) **54**, 275–285 (1951).

In part I of this paper [Trans. Amer. Math. Soc. **70**, 160–176 (1951); these Rev. **12**, 718], R. H. Cameron and the author "investigated the structure of the class of additive functionals of class L_2 on the Wiener space C and obtained certain other results concerning functionals additive and merely measurable on C. Recently Cameron conjectured that every functional additive and measurable on C was of class $L_2(C)$, and hence that the characterization and representation theory previously developed applied to all functionals additive and measurable on C."

In the present paper the author proves Cameron's conjecture. The proof is based upon the following theorem which the author proves concerning the Fourier-Hermite expansion of $e^{iF(x)}$ where $F(x)$ is measurable and additive on C, namely that

$$e^{iF(x)}=\text{L.I.M.}_{n\to\infty} A_0 \sum_{m_1,\cdots,m_n=0}^{n}\left\{\prod_{j=1}^{n}[(m_j!)^{-\frac{1}{2}}(i2^{-\frac{1}{2}}F(\beta_j))^{m_j}]\right\}\cdot\Psi_{m_1,\cdots,m_n}(x),$$

where $A_0=\int_0^v \cos(F(x))d_w x\neq 0$ and where $\beta_j(t)=\int_0^t\alpha_j(u)du$ and $\{\alpha_j(t)\}$ is the complete orthonormal set used in the definition of the Fourier-Hermite functionals. By use of Bessel's inequality the author shows under the same hypotheses on F that (1) $\sum_1^\infty[F(\beta_j)]^2$ converges and that for almost all $x\in C$, (2) $F(x)=\sum_1^\infty F(\beta_j)\int_0^t\alpha_j(t)\tilde{d}x(t)$. By a result due to A. Kolmogoroff [Math. Ann. **99**, 309–319 (1928)] the convergence almost everywhere of the series (2) follows from the convergence of (1). But the convergence of (1) also implies that the series in (2) converges in the $L_2(C)$ mean, and this yields Cameron's conjecture. This result enables the author to simplify the proofs of some of the results in part I and to strengthen other results. *W. T. Martin.*

Taldykin, A. T. Systems of elements of a Hilbert space and series formed from them. Mat. Sbornik N.S. **29(71)**, 79–120 (1951). (Russian)

Let $\{\varphi_r\}$ be a sequence of elements of a Hilbert space, Φ the matrix $\|(\varphi_r,\varphi_s)\|=\|\varphi_{rs}\|$, Φ_n the $n\times n$ upper lefthand corner of Φ, $\|\varphi_{rs}\|=\Phi_n^{-1}$. Let λ_1^n and λ_n^n denote the lowest and highest eigenvalues of Φ_n, $\lambda_1=\lim\lambda_1^n$, and $\lambda'=\lim\lambda_n^n$ is finite if Φ is bounded. Systems of elements are classified according to the properties of Φ; also a system is called minimal if removal of any element reduces the subspace spanned by the system. For a minimal system there exists a biorthogonal system $\{\varphi^r\}$. The unique biorthogonal system lying in the subspace $[\varphi_r]$ spanned by $\{\varphi_r\}$ is called the allied system. If $\lambda_1>0$ the allied system has a bounded matrix and the system is called strongly minimal. If $\lambda_1=0$ the system, if minimal, is called weakly minimal: this implies that 0 is not in the point spectrum of Φ.

A strongly minimal system with bounded matrix (in the operator sense) is called normal; such systems have many properties generalising those of orthonormal systems. The spaces $[\varphi_r]$ and $[\varphi^r]$ coincide. If $(f,\varphi_r)=A_r$, $(f,\varphi^r)=A^r$, and $\hat{f}$ is the projection of f on $[\varphi_r]$, then

$$\frac{1}{\lambda'}\sum|A_r|^2\leqq\|\hat{f}\|^2\leqq\frac{1}{\lambda_1}\sum|A_r|^2,\quad \lambda_1\sum|A^r|^2\leqq\|\hat{f}\|^2\leqq\lambda'\sum|A^r|^2.$$

A necessary and sufficient condition for $\{c_r\}$ to be the set of components of an element, or for $\sum c_r\varphi_r$ to be strongly convergent, is $\sum|c_r|^2<\infty$. More generally, if $\{\varphi^r\}$ is allied to a strongly minimal system, $\sum|A^r|^2$ is convergent, and $\sum A_r\varphi^r$ converges strongly to $\hat{f}$ if $\sum|A_r|^2$ is convergent. $a_n^r=\sum\varphi_n^{rs}A_s$ is the coefficient of φ_r in the best approximation to f by a linear combination of $\varphi_1,\cdots,\varphi_n$. If $\{\varphi_r\}$ is strongly minimal, a_n^r tends weakly to A^r as $n\to\infty$; if, in particular, Φ is the matrix of a selfadjoint operator, convergence is strong. If $\{\varphi_r\}$ is strongly minimal and $\sum|\varphi_{rk}|^2<\infty$ for all k, then $\|(\varphi^r,\varphi^s)\|=\|\varphi^{rs}\|$ is a^r left inverse to Φ and $\sum\varphi_{rs}A^r=A_s$ for any f. If f is such that $\sum|A_r|^2<\infty$, then $A^r=\sum\varphi^{rs}A_s$ and, for any g, $\sum A_r(\varphi^r,g)$ converges to $(\hat{f},g)$. Further theorems are given concerning convergence of series of these types under varying hypotheses on Φ and on the system. *J. L. B. Cooper* (Cardiff).

Gelbaum, B. A nonabsolute basis for Hilbert space. Proc. Amer. Math. Soc. **2**, 720–721 (1951).

The Hilbert space $L^2(-\pi,\pi)$ is considered. By a basis $\{f_n;g_n\}$ $(n=0,1,2,\cdots)$ is meant a pair of sequences $\{f_n\}$, $\{g_n\}$ from the Hilbert space such that $f=\sum_{n=0}^\infty(f,g_n)f_n$ for every f in the space, and $(f_m,g_n)=\delta_{mn}$. The basis is called absolute if the expansion of f is unconditionally convergent to f for every f. In this paper there is exhibited a basis which is not absolute. The sequences are

$$\{f_n\}:\ \frac{c}{\sqrt{2}}|x|^\alpha\varphi_0,\ c|x|^\alpha\varphi_1,\ c|x|^{-\alpha}\varphi_2,\ c|x|^\alpha\varphi_3,\ c|x|^{-\alpha}\varphi_4,\ \cdots,$$

$$\{g_n\}:\ \frac{c}{\sqrt{2}}|x|^{-\alpha}\varphi_0,\ c|x|^{-\alpha}\varphi_1,\ c|x|^\alpha\varphi_2,\ c|x|^{-\alpha}\varphi_3,\ c|x|^\alpha\varphi_4,\ \cdots,$$

where $0<\alpha<\frac{1}{2}$, $c=(\pi)^{-\frac{1}{2}}$, and $\{\varphi_n\}$ is the sequence $1,\cos x,\sin x,\cos 2x,\sin 2x,\cdots$. That $\{f_n;g_n\}$ is a basis was proved by M. S. Al'tman [Doklady Akad. Nauk SSSR (N.S.) **69**, 483–485 (1949); these Rev. **11**, 525]. The present paper demonstrates that the basis is not absolute.
A. E. Taylor (Los Angeles, Calif.).

***Fuglede, Bent. A commutativity theorem for normal operators.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 221–227. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

An exposition is given of the author's affirmative answer to the following question asked by J. von Neumann. Is it true that if a bounded linear operator B (in a Hilbert space) commutes with a normal operator N, then B commutes with N^* also. See Fuglede [Proc. Nat. Acad. Sci. U. S. A. **36**, 35–40 (1950); these Rev. **11**, 471]; cf. also Halmos [Acta Sci. Math. Szeged **12B**, 153–156 (1950); these Rev. **11**, 600]. *K. Yosida* (Nagoya).

***Hamburger, Hans. Non-symmetric operators in Hilbert space.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 67–112. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Eight lectures are devoted to a theory of general bounded operators in a Hilbert space $\mathfrak{H}$, with the general objective of extending the Jordan canonical decomposition theory to them. Lecture I. Let C be a completely continuous operator with range $\mathfrak{R}$, simple nonzero eigenvalues λ_n, $|\lambda_n|\geqq|\lambda_{n+1}|\to 0$ and eigenvectors $\{\varphi_n\}$. Let $\{\psi_n\}$ be the eigenvectors of C^*, normalised so that $(\varphi_n,\psi_m)=\delta_{nm}$; let $\{\varphi_n\}$, $\{\psi_n\}$ span the

subspaces $\mathfrak{S}$, $\mathfrak{S}^*$ respectively, let $\mathfrak{N}=\mathfrak{S}^{*\perp}$ (orthogonal complement), $\mathfrak{N}^*=\mathfrak{S}^{\perp}$. Then $C(\mathfrak{S})\subset\mathfrak{S}$, $C(\mathfrak{N})\subset\mathfrak{N}$, $\|C^n x\|^{1/n}\to 0$ as $n\to\infty$ for all x in $\mathfrak{N}$; this last property is expressed by calling C quasinilpotent in $\mathfrak{N}$. It is shown that if $\mathfrak{N}\cap\mathfrak{S}=0$, then $\mathfrak{S}\oplus\mathfrak{N}=\mathfrak{H}$, and if $C^*(\mathfrak{N}^*\cap\mathfrak{S}^*)=0$, then $\mathfrak{N}\subset\mathfrak{N}\oplus\mathfrak{S}$. Conditions are given for $\mathfrak{N}\cap\mathfrak{S}=0$ or $C(\mathfrak{N}\cap\mathfrak{S})=0$ to hold. In all cases the corresponding *-properties, obtained by writing the adjoint, also follow. Lecture II is devoted to the construction of a C with simple eigenvalues for which $\mathfrak{N}\cap\mathfrak{S}$ is of infinite dimension, $C(\mathfrak{N}\cap\mathfrak{S})\neq 0$ and the corresponding properties hold.

Lecture III begins the study of operators which generalise quasinilpotents. If A is bounded and $\mathfrak{D}_n$ is the set of all x for which $\|A^n x\|^{1/n}\to 0$, $\mathfrak{D}_n{}^*$ that for which $\|A^{*n}x\|^{1/n}\to 0$, then, if either $\mathfrak{D}_n$ or $\mathfrak{D}_n{}^*$ is dense in $\mathfrak{H}$, A is called an N transformation; if both $\mathfrak{D}_n$ and $\mathfrak{D}_n{}^*$ are dense, a perfect N transformation. If A is a perfect N transformation, A and A^* have no nonzero eigenvalues. Sufficient conditions for a transformation to be a perfect N transformation and an example of one which is not quasinilpotent are given. The relative adjoint of A with respect to a subspace $\mathfrak{N}$ is defined by $A^{[*]}y\in\mathfrak{N}$, $(Ax,y)=(x,A^{[*]}y)$ if $x\in\mathfrak{N}$, $y\in\mathfrak{N}$, and perfect N transformations with respect to subspaces are considered. Lecture IV generalises the Jordan decomposition. $\mathfrak{X}^n$ ($\mathfrak{X}^0=\mathfrak{X}$), $\mathfrak{X}^{*n}$, denote the null manifolds of A^n and A^{*n}, $\mathfrak{R}^n$, $\mathfrak{R}^{*n}$ their ranges, $\mathfrak{X}^\infty=\bigcup\mathfrak{X}^n$, $\mathfrak{R}^\infty=\bigcap\mathfrak{R}^n$ etc. $\mathfrak{R}_\lambda$ denotes the range of $A_\lambda=A-\lambda I$. The operator A is called of type τ if $\overline{\mathfrak{X}}^\infty=\mathfrak{H}$, $\mathfrak{R}^\infty=0$, of type τ_0 if in addition $\mathfrak{R}^n$ is closed for all n. A Jordan chain of order k of A is a set $\{\xi^n\}$, $n=1,\cdots,k$, such that $A\xi^1=0$, $A\xi^n=\xi^{n-1}$, $2\leqq n\leqq k$, ξ^k non-$\in\mathfrak{R}$. If k is infinite and the last condition removed we have an infinite Jordan chain. It is shown that any τ transformation is a perfect N transformation, and that if A is a τ_0 transformation $\mathfrak{H}$ can be spanned by an infinite set of finite Jordan chains.

Lecture V. Let A be a bounded linear transformation, λ and eigenvalue, $\mathfrak{E}$ the corresponding eigenmanifold; then λ is called an improper eigenvalue with respect to a subspace $\mathfrak{M}$ if $\mathfrak{E}\subset\mathfrak{M}$, $A(\mathfrak{M})\subset\mathfrak{M}$, $A_\lambda(\mathfrak{M})$ is everywhere dense in $\mathfrak{M}$. It is shown that λ is not an eigenvalue of $A^{[*]}$, but is in the residual spectrum for A^*. The eigenvalues of A improper with respect to $\mathfrak{M}$ are called the coresidual spectrum of A with respect to $\mathfrak{M}$. If $\mathfrak{N}=\overline{\mathfrak{D}}_m$, and $\mathfrak{E}\subset\mathfrak{N}$, then λ is an improper eigenvalue of A with respect to $\mathfrak{N}$. If $\mathfrak{M}\subset\mathfrak{N}$ and $A(\mathfrak{M})\subset\mathfrak{M}$, A is called a coresidual N transformation with respect to $\mathfrak{M}$, A^* a residual N transformation.

Lecture VI. A is said to be of 1st kind if its range is closed, λ of first kind if the range of A_λ is closed. If $\mathfrak{R}_\lambda=\mathfrak{H}$, λ is an interior point of the set of points of 1st kind. If A is of first kind, $\mathfrak{N}=\overline{\mathfrak{X}}^\infty=\mathfrak{H}$, and $\mathfrak{X}$ is of rank 1, then A and A^* are respectively residual and coresidual N transformations in $\mathfrak{H}$ and an infinite Jordan chain, spanning $\mathfrak{H}$, exists for A. Lecture VII discusses operators defined by an infinite Jordan chain which is a stationary sequence, i.e. (ξ^m,ξ^n) is a function of $(m-n)$ only. Lecture VIII discusses general questions which arise in attempting to express $\mathfrak{H}$ as the span of Jordan chains for an operator A of 1st kind, and gives various conditions under which this is possible.

J. L. B. Cooper (Cardiff).

von Neumann, Johann. Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes. Math. Nachr. **4**, 258–281 (1951).

Let A be a bounded linear operator in the unitary space U and let S be a set of complex numbers. If for every rational function f with $|f(\lambda)|\leqq 1$ for $\lambda\in S$ we have $f(A)$ existing as an everywhere defined and bounded operator with bound $|f(A)|\leqq 1$, then S is said to be a spectral set of A. It is shown that the closed unit circle $[\lambda;\ |\lambda|\leqq 1]$ is a spectral set of A if and only if $|A|\leqq 1$. The proof makes use of the theory of E-functions introduced by I. Schur. Of the many interesting consequences we mention the following. If α is complex and $\beta>0$ the set of all λ with $|\lambda-\alpha|\leqq\beta$ ($|\lambda-\alpha|\geqq\beta$) is a spectral set of A if and only if $|(A-\alpha I)|\leqq\beta$ ($|(A-\alpha I)^{-1}|\leqq\beta^{-1}$). The intersection of all spectral sets of A is the spectrum of A. Either one of the following statements is necessary and sufficient for A to be unitary. The set of λ with $|\lambda|\leqq 1$ and the set of λ with $|\lambda|\geqq 1$ are spectral sets of A. The set of λ with $|\lambda|=1$ is a spectral set of A. A similar result holds for hermitian operators. The operator A is normal if and only if it has an absolute minimal spectral set. Another condition equivalent to normality is that the intersection of two spectral sets is a spectral set.

N. Dunford (New Haven, Conn.).

Rothe, E. H. Critical points and gradient fields of scalars in Hilbert space. Acta Math. **85**, 73–98 (1951).

Sia H uno spazio hilbertiano, V un insieme aperto e convesso di H, S la sua frontiera. Detto $I(x)$ un funzionale reale definito in una regione aperta V' di H, contenente $V+S$, ivi differenziabile, con $dI=(G(x),\delta x)$, dove $G(x)$ è un operatore completamente continuo, l'autore considera i punti critici del funzionale $i(x)=\frac{1}{2}\|x\|^2+I(x)$, cioè le soluzioni dell'equazione (1) $x+G(x)=0$. Un tale punto critico è detto non degenere se verifica certe ipotesi che, nel caso di una funzione i di n variabili, si riducono a quella dell'hessiano di i non nullo. Se $i(x)$ è tale che i suoi punti critici contenuti in V sono tutti non degeneri e se su S non cadono punti critici, in V cade al più un numero finito di punti critici $a_1,a_2,\cdots,a_\nu$. Posto: $dG(x)=(L(x),\delta x)$ con $L(x)$ completamente continuo, e detto r_k il numero degli autovalori del parametro λ minori di -1 per l'equazione $\lambda\delta x=(L(a_k),\delta x)$, l'indice del punto a_k come soluzione della (1) è dato da $(-1)^{r_k}$ e, posto: $M^i=\sum_{k=1}^{\nu}\delta^i{}_{n_k}$ ($\delta_j{}^i$ simbolo di Kronecker) si ha: $\gamma=\sum_i(-1)^i M_i$, essendo γ il grado topologico nell'origine di H della trasformazione $y=x+G(x)$.

Da questo risultato si possono dedurre vari criteri di esistenza e anche di unicità di punti critici. L'autore applica uno di tali risultati allo studio di certi sistemi di equazioni integrali non lineari più generali di quelli considerati da M. Golomb [Publ. Math. Univ. Belgrade **5**, 52–83 (1936)].

C. Miranda (Napoli).

Vainberg, M. M. On weak continuity of functionals and of their gradients. Doklady Akad. Nauk SSSR (N.S.) **78**, 841–844 (1951). (Russian)

The author calls the operator $F(x)$ (or functional $f(x)$) on the real Hilbert space "weakly continuous" at x_0 if $F(x_n)\to F(x_0)$ strongly ($f(x_n)\to f(x_0)$) whenever $x_n\to x_0$ weakly. The functional $f(x)$ has the gradient $F(x)$ if (for fixed x) $f(x+h)-f(x)-(F(x),h)=o(\|h\|)$ as $\|h\|\to 0$. The main result proved is: $F(x)=\operatorname{grad} f(x)$ is weakly continuous in $\|x\|\leqq r$ if and only if $F(x)$ is completely and uniformly continuous in $\|x\|\leqq r$. *M. Golomb* (Lafayette, Ind.).

Glazman, I. M. On the theory of singular differential operators. Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 102–135 (1950). (Russian)

A theory of selfadjoint differential equations of order $2n$ is developed by using the theory of symmetric operators in Hilbert space. Preliminary results are given for abstract

operators A, B. Defining the deficiency of a subspace to be the dimension of its orthogonal complement, and $\mathrm{def}_\lambda A$ to be the deficiency of $(A-\lambda I)\mathfrak{H}$, it is shown that if A, B are closed linear operators with dense domains for which 0 is a regular point with finite deficiency index for each of A, B, then AB is closed with dense domain and

$$\mathrm{def}_0 AB=\mathrm{def}_0 A+\mathrm{def}_0 B.$$

The symbol

$$l=p_n-D\{p_{n-1}D-D[p_{n-2}D^2-\cdots-D(p_1D^{n-1}-Dp_0D^n)]\},$$

defines a selfadjoint differential operation if the p's are sufficiently often differentiable. If, in an interval (a, b), the p_k are measurable and (1) $p_0(t)$ and $p_k(t)$ $(k=1, \cdots, n)$ are in $L(a', b')$ whenever $a<a'<b'<b$, then l is called a quasi-differential operation. Defining $D^{[k]}=D^k$ $(k=0, \cdots, n-1)$, $D^{[n]}=p_0D^n$, $D^{[n+k]}=p_kD^{n-k}-DD^{[n+k-1]}$ $(k=1, \cdots, n)$, we have $l=D^{[2n]}$. Then l is defined for all functions g with absolutely continuous quasiderivatives $g^{[k]}(t)$ $(k\leqq 2n-1)$. If (a, b) is finite and (1) holds with $a'=a$, $b'=b$, then l is called regular; if it fails to hold at a, b or both, or if the interval is infinite, l is called singular (at one end, or at both ends). (a, b) is taken to be $(0, \infty)$ in the case with one singular end, $(-\infty, \infty)$ for two singular ends.

A theory of solutions of quasidifferential equations $l(y)=g(t)$ with Cauchy initial conditions, similar in most ways to the classical theory, with quasiderivatives replacing derivatives, is developed. It involves, in particular, a Lagrangian identity, $l(g)h-gl(h)=(d/dt)[g, h]_t$, the latter being a bilinear form in g, h.

From the operation l are defined operators on the Hilbert space $L^2(a, b)$. The maximal such operator has domain $\mathfrak{D}$ consisting of all $g(t)$ such that $g^{[k]}(t)$ is absolutely continuous for $0\leqq k\leqq(2n-1)$, and $g^{[2n]}(t)\in L^2$. The minimal operator, in $L^2(0, \infty)$, has as domain those elements of $\mathfrak{D}$ which vanish outside a finite interval and satisfy $g^{[k]}(0)=0$, $k\leqq(2n-1)$. These operators are both defined by being equal to $l(g)$ in their domain; the closure of the minimal operator, which is symmetric, is denoted by L: L^* is the operator with maximal domain. It is shown that $n\leqq\mathrm{def}\, L\leqq 2n$, and that if $m(\lambda)$ is the number of independent solutions of $l(u)-\lambda u=0$ which lie in $L^2(0, \infty)$, then $m(\lambda)=\mathrm{def}\, L$ for all nonreal λ, $m(\lambda)\leqq\mathrm{def}\, L$ for all real λ. W. Windau [Math. Ann. 83, 256–279 (1921)], for $2n=4$, and D. Schin [Mat. Sbornik 7(49), 479–532 (1940); these Rev. 2, 51], for the general case, have claimed to show that def L must be either n or $2n$, by a method generalising the classical results of H. Weyl [Math. Ann. 68, 220–269 (1910)]; however, examples are given to show that def L can take any value between n and $2n$. It is also shown that if L is defined for $L^2(-\infty, \infty)$, as the closure of the operator equal to l on the part of $\mathfrak{D}$ consisting of functions vanishing outside a finite interval, then, if L^+, L^- are the operators on $L^2(0, \infty)$, $L^2(-\infty, 0)$ corresponding to L, $\mathrm{def}\, L=\mathrm{def}\, L^+ +\mathrm{def}\, L^- -2n$.

Selfadjoint extensions of L are discussed, firstly for $(0, \infty)$. If def $L=m$, and $u_k(t; \lambda)$, $1\leqq k\leqq m$, are orthonormal solutions of $l(u)-u=0$ for a fixed nonreal λ, then, in a well-known way, the extensions L_θ of L correspond to unitary matrices $\theta=\|\theta_{ij}\|$; they have as domains the sets $\mathfrak{D}_\theta=\mathfrak{D}_L\oplus\mathfrak{G}_\theta$, where G_θ is the linear hull of the functions

$$w_i^\theta(t; \lambda)=u_i(t; \lambda)+\sum_k\theta_{ik}u_k(t; \lambda).$$

It is further shown that $\mathfrak{D}_L$ consists of those $g\in\mathfrak{D}$ which satisfy $[g, w]_0^\infty=0$ for all $w\in\mathfrak{D}$, $\mathfrak{D}_\theta$ of those which satisfy $[g, w_i^\theta]_0^\infty=0$ $(1\leqq i\leqq m)$. If def $L=n$, the upper limit in this bracket is automatically 0. For a selfadjoint extension to be a real operator, θ must be a symmetric matrix; and if its resolvent operator can be expressed as an integral operator with continuous kernel $K(t, s, \lambda)$, then K is symmetric in t and s. The resolvent operator of any selfadjoint extension for any regular point λ can be expressed as an integral operator: $R_\lambda g(t)=\int_0^\infty K(t, s; \lambda)g(s)ds$, and that $K(t, s; \lambda)$ is of summable square in each of t and s separately (Carleman kernel), while if def $L=2n$ it is of summable square in the two variables (Hilbert-Schmidt kernel).

In order that def $L=2n$, it is necessary that $m(\lambda)=2n$ for all real λ, and sufficient that this hold for one real λ. Similar results are obtained for the case of two singular ends.

J. L. B. Cooper (Cardiff).

Fuglede, Bent, and Kadison, Richard V. On a conjecture of Murray and von Neumann. Proc. Nat. Acad. Sci. U. S. A. **37**, 420–425 (1951).

Let R be a ring of operators and S an arbitrary subset of R. Denote by S_R' the set of those elements of R which commute with every element of S. Murray and von Neumann [Ann. of Math. (2) **37**, 116–229 (1936)] have called a ring of operators normal if $(S_R')_R'=S$ for every subring S of R. They conjectured that all factors of type II are non-normal. This conjecture is proved in the present note. In fact it is shown (Theorem 3) that if M is a factor of type II_1 which is not approximately finite and A a maximal approximately finite subfactor of M, then $(A_M')_M'$ contains A properly. The case of an arbitrary factor of type II is reduced to the above. In order to prove this theorem the authors derive some other interesting results. Theorem 1 gives a condition for an operator to be small in the metric defined by the (relative) trace. Theorem 2 deals with the family of approximately finite subfactors under set-inclusion. In particular, the existence of maximal approximately finite subfactors is proved. It seems to the reviewer that the methods of this note should work also in the case where M is not a factor but an arbitrary ring of operators of type II. Although this case seems likely to be reducible to that of factors by the use of von Neumann's reduction theory, a direct extension of the present authors' method would be much more elementary.

F. I. Mautner (Baltimore, Md.).

Fuglede, Bent, and Kadison, Richard V. On determinants and a property of the trace in finite factors. Proc. Nat. Acad. Sci. U. S. A. **37**, 425–431 (1951).

In this note a theory of the analogue of the absolute value of the determinant is outlined in a factor M of finite type. Suppose $X\in M$ and X^{-1} exists and is an element of M. Then the authors call X regular. One has $X=UH$ uniquely with U unitary, H self-adjoint, positive and regular and both in M. The determinant $\Delta(X)$ is defined by

$$\Delta(X)=\Delta(H)=\exp\left[T(\log H)\right]=\exp\left[\int\log\lambda dD(E_\lambda)\right],$$

where T denotes the relative trace in M and $H=\int\lambda dE_\lambda$. Thus $\Delta(U)$ is defined to be 1 for every unitary $U\in M$ and hence if M is finite-dimensional $\Delta(X)$ is the (nth root of the) absolute value of the ordinary determinant. The authors state properties of $\Delta(X)$ and outline the proofs. As an example we state the result that $\Delta(X)$ does not exceed the spectral radius of X and is continuous in the uniform topology (on the regular elements of M). This is used to prove theorem 2: The relative trace $T(A)$ of an arbitrary element A of M is in the convex hull of the spectrum of A. In particular $T(A)=0$ if A is generalized nilpotent. Theorem 3 asserts that certain properties determine $\Delta(X)$ uniquely.

In the last paragraph the authors indicate the reasons which compel them to consider only positive-valued determinants and also discuss the possibilities of extending $\Delta(X)$ to the singular elements of M. *F. I. Mautner.*

Kadison, Richard V. Isometries of operator algebras. Ann. of Math. (2) **54**, 325–338 (1951).

Let $C(X)$ denote the Banach algebra of all continuous functions on the compact Hausdorff space X. The well-known results of Banach [Théorie des opérations linéaires, Warsaw, 1932] and Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] concerning isometric mappings of such Banach algebras give the following theorem: Every isometric linear mapping of $C(X)$ onto a second such Banach algebra $C(Y)$ is equal to an isometric algebra isomorphism of $C(X)$ onto $C(Y)$ followed by multiplication with a function in $C(Y)$ whose absolute value is constantly equal to one. The present paper contains a non-commutative extension of this result in which the algebras involved are C^*-algebras (i.e. uniformly closed self-adjoint algebras of bounded operators on a Hilbert space). More precisely, every isometric linear mapping of one C^*-algebra $\mathfrak{A}$ onto a second C^*-algebra $\mathfrak{A}'$ is equal to a Jordan $*$-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$ followed by multiplication with a unitary operator in $\mathfrak{A}'$. Conversely, every such mapping is an isometry. A Jordan $*$-isomorphism (called a C^*-isomorphism by the author) is a one-to-one linear mapping which preserves the $*$-operation and the Jordan product $\{AB\}=AB+BA$. The proof of the general theorem is independent of the commutative case and involves the following characterization of extreme points on the unit sphere of a C^*-algebra: U is an extreme point of the unit sphere of $\mathfrak{A}$ if, and only if, U is a partially isometric operator [Murray and von Neumann, Ann. of Math. (2) **37**, 116–229 (1936)] such that $(I-UU^*)\mathfrak{A}(I-U^*U)=(0)$.

Using a decomposition theorem for rings of operators (weakly closed C^*-algebras [cf. Murray and von Neumann, loc. cit]) due to Kaplansky [Ann. of Math. (2) **53**, 235–249 (1951); these Rev. **13**, 48] and results of Jacobson and Rickart [Trans. Amer. Math. Soc. **69**, 479–502 (1950); these Rev. **12**, 387] on Jordan homomorphisms, the author shows that every Jordan $*$-isomorphism of a ring of operators onto a C^*-algebra is the sum of a $*$-isomorphism and a $*$-anti-isomorphism. A corollary is that any Jordan $*$-isomorphism of a Murray-von Neumann factor [loc. cit.] is either a $*$-isomorphism or $*$-anti-isomorphism.

If two extreme points of the unit sphere of a C^*-algebra $\mathfrak{A}$ are called "equivalent" provided one is the image of the other under an isometry of $\mathfrak{A}$ onto itself, then the extreme points fall into equivalence classes. By examination of these extreme point classes, the author is able to characterize factors of types I_n, II_1 and III in terms only of their Banach space structure. Due to incomplete knowledge of extreme point classes, similar characterizations for factors of type I_∞ and II_∞ are not known. *C. E. Rickart.*

Edwards, R. E. Multiplicative norms on Banach algebras. Proc. Cambridge Philos. Soc. **47**, 473–474 (1951).

The theorem that a normed algebra A over the reals satisfying $\|xy\|=\|x\|\cdot\|y\|$ for all x, y in A is the set of real numbers, complex numbers, or real quaternions, is due to Mazur [C. R. Acad. Sci. Paris **207**, 1025–1027 (1938)]. The complex commutative case was also treated by I. Gelfand and the reviewer. The author considers a real or complex commutative A and proves the above result under the weakened hypothesis that $\|x^{-1}\|=\|x\|^{-1}$ for all regular x. The proof is based on a lemma that in A (unless it is real and one-dimensional) the set A_ρ of elements x with $\|x\|\geqq\rho$, $\rho>0$, is connected. From this it follows that the set $R_\rho=A_\rho\cap R$ (R the set of regular elements in A) is both open and closed in A_ρ, hence is identical with A_ρ. It is indicated how the proof of the theorem may be extended to certain types of topological algebras. *E. R. Lorch.*

Williams, Charles W. Recurrence and incompressibility. Proc. Amer. Math. Soc. **2**, 798–806 (1951).

Let X be a separable metric space and let T be an abelian multiplicative topological group generated by some compact neighborhood of e, the identity in T. Let T act as a transformation group on X so that to $x\varepsilon X$, $t\varepsilon T$ there is assigned a point xt of X in such a way that $xe=x$, xt is continuous on the product space $X\times T$ and such that $(xt)s=x(ts)$ for t, $s\varepsilon T$, $x\varepsilon X$. Gottschalk and Hedlund [Trans. Amer. Math. Soc. **65**, 348–359 (1949); these Rev. **10**, 718] have given necessary and sufficient conditions in terms of an "incompressibility" property that every point of X be periodic under T and that every point of X be recurrent under T. We shall say that the dynamical system (X, T) has property A (B) $\{C\}$ if when M is a subset (closed subset) {compact subset} of X and S is a replete semigroup in T such that $MS\subset M$, then $M-MS$ is a set of the first category. The author shows that (1) the periodic points of X form a residual set when, and only when, (X, T) has property A and that (2) if X is complete, the recurrent points of X form a residual set when, and only when, (X, T) has property B.

Other results give decompositions (ignoring a set of first category) of X into the sum of two sets, one of which is the set of recurrent points of X. For example, it shown that (3) if X is complete, then the sum of the recurrent points and wandering points of X form a residual set and (4) if X is locally compact and complete and if (X, T) has property C, then the recurrent points and the unstable points of X form a residual set. *W. R. Utz* (Columbia, Mo.).

Saito, Toshiya. Examples of ergodic dynamical systems. Kōdai Math. Sem. Rep. **1951**, 21–25 (1951).

Let a flow on the torus be defined by differential equations $\dot{x}=X(x, y)$, $\dot{y}=Y(x, y)$, and let $\int\int\rho dxdy$ be an invariant integral, where ρ is positive and ρ, X, and Y are of class C'. It is stated that the flow is ergodic if and only if X and Y have no common zero and the ratio of the mean value of ρX on the torus to that of ρY is irrational. (The author refers to a forthcoming paper of his in J. Math. Soc. Japan.) This criterion is then applied to four systems: (1) two simple harmonic motions at right angles, (2) planetary motion in special relativity, (3) a particle moving on a paraboloid in a gravitational field, and (4) a particle attracted by two fixed centers of gravitation. It is shown that system (1) is ergodic except when the ratio of the periods is rational, and that systems (2), (3), and (4) are ergodic except when the energy E and a momentum constant h satisfy certain explicitly stated relations. In each case the exceptional set forms a set of first category in the domain of (E, h) space to which these constants must be restricted. *J. C. Oxtoby.*

Fomin, S. On dynamical systems in a space of functions. Ukrain. Mat. Žurnal **2**, no. 2, 25–47 (1950). (Russian)

This paper contains mainly detailed proofs and discussions of results announced earlier by the author [Doklady Akad. Nauk SSSR (N.S.) **67**, 435–437 (1949); these Rev. **11**, 117]. Extensions of these results to the dynamical system

$\Omega=(X, S_t)$ where X is the space of all real-valued functions $f(x)$ on the line and $S_t f(x)=f(x+t)$ for every t are also discussed. *Y. N. Dowker* (Manchester).

Fomin, S. On dynamical systems with a purely point spectrum. Doklady Akad. Nauk SSSR (N.S.) **77**, 29–32 (1951). (Russian)

Let L be a countable subgroup of the discrete additive group of real numbers. Let X be the character group of L. Let S_t be a 1-parameter group of transformations defined on X by $S_t\chi(\lambda)=e^{\Omega t}\chi(\lambda)$, $\lambda\varepsilon L$, $\chi(\lambda)\varepsilon X$. Then (X, S_t) is called a canonical dynamical system. It is known [Halmos and von Neumann, Ann. of Math. (2) **43**, 332–350 (1942); these Rev. **4**, 14] that not only is a canonical system an ergodic dynamical system with a pure point spectrum but that every ergodic dynamical system with a pure point spectrum is isomorphic to a canonical dynamical system. In this paper the author studies some of the topological properties of dynamical systems with a pure point spectrum homeomorphic to a canonical system. Among the results obtained is the following: A necessary and sufficient condition for a dynamical system Ω to be homeomorphic to a canonical dynamical system is that Ω be compact, strongly ergodic and stable in the sense of Liapounoff. Here stability in the sense of Liapounoff means the equi-continuity of the 1-parameter group of homeomorphisms on Ω. It follows that there exists a complete set of continuous proper functions on every compact, strongly ergodic dynamical system stable in the sense of Liapounoff. A sufficient condition is also given for a dynamical system with a pure point spectrum to have a continuous mapping onto a canonical system with the same spectrum. This condition is given in terms of a relaxed stability condition. The results obtained in the paper are used to study trajectories on the surface of the torus. Proofs are only indicated. *Y. N. Dowker.*

Calculus of Variations

***Picone, Mauro. Lezioni di analisi funzionale.** Dispense Universitarie Tumminelli, Rome, undated. ii+576 pp. Lire 1,450.

This work is principally devoted to the calculus of variations. In an introductory chapter, the author discusses abstract sets and Hausdorff spaces. In the statement of the neighborhood axioms for a Hausdorff space, the requirement is inserted that the space be dense-in-itself. In discussing examples, the author does not make clear the importance of the topological structure. For instance, he mentions examples of spaces of a denumerable infinity of dimensions, but refers to an erroneous definition of dimension. In the final section of the introduction, the concepts of continuity, semi-continuity, bounded variation, absolute continuity, and equicontinuity are defined and the theorem of Ascoli-Arzelà is proved.

Chapter I gives some theorems on the existence of a minimum in the calculus of variations, under rather restrictive conditions on the class of curves admitted or on the integrand. Section 3 of this chapter discusses the space of rectifiable curves, Hilbert's theorem on compactness, and the integral of Weierstrass of the form $\int f(x)ds$. Chapter II is devoted to metric spaces. In their definition, the symmetry of the distance function seems to have been overlooked, but it is used here and there. Pairs of points whose distance is infinite are admitted, and the space is again required to be dense-in-itself. A continuum is defined as a closed set E every pair of whose points can be joined by an ϵ-chain in E with ϵ arbitrarily small. The equivalence of the space L_2 with classical Hilbert space is discussed in detail. Homogeneous polynomials of degree n in L_2 are defined by an integral formula with a kernel which is a function of n variables, of integrable square. This is a more restrictive definition than the one commonly used in functional analysis, and results in the theorem that polynomials are continuous in terms of weak convergence in L_2. So a polynomial has a maximum and a minimum in a sphere in L_2. The author derives the Euler equation for the minimizing function in the sphere, and for the minimizing function on the surface of the sphere.

In Chapter III the elementary theory of the calculus of variations for nonparametric simple integrals in the plane is discussed. The author mentions the minimum problem for the second variation, but does not use this basic idea of Bliss effectively to obtain simple proofs of the necessary conditions. The sufficiency theorems are proved by expansion methods.

Chapter IV makes a fresh start and discusses only the first variation and the Euler equation, but more general problems of the calculus of variations are admitted, including parametric problems, problems with variable end points, a very special case of the problem of Mayer, the problem of Lagrange, and the simplest double integral problem. A proof of the multiplier rule for the problem of Lagrange is given only for the case when the integrand is a function of x, y_1, y_2, y_1', y_2', and the side condition has the form $\varphi(x, y_1, y_2)=0$. Reference is made to Bolza's book for the general case. Chapter V discusses some applications of the calculus of variations to the theory of elasticity, including the determination of equilibrium positions and the theory of vibration. The book is lithographed from a handwritten manuscript. *L. M. Graves* (Chicago, Ill.).

Magenes, Enrico. Condizioni sufficienti per il minimo relativo in certi problemi di Mayer. Rend. Sem. Mat. Univ. Padova **20**, 78–98 (1951).

For the Mayer problem of minimizing $\mathcal{J}=y_2(x_2)$ in a class of admissible curves satisfying the single differential equation $\varphi(x, y_1, y_1', y_2)-y_2'=0$ and the end-conditions $x_1-a=0=x_2-b=y_2(x_1)-y_{2,a}$, where a, b, $y_{2,a}$ are constants, the author shows that if an admissible curve $\bar{\Gamma}$: $\bar{y}_1(x)$, $\bar{y}_2(x)$, $a\leqq x\leqq b$, satisfies the usual conditions I, $II_{N'}$, III', IV' then there exists a neighborhood (ρ) of $\bar{C}$: $\bar{y}_1(x)$ in (x, y_1)-space such that if Γ: $y_1(x)$, $y_2(x)$, $a\leqq x\leqq b$, is an admissible curve with C: $y_1(x)$ in (ρ) then $y_2(x_2)\geqq\bar{y}_2(x_2)$, and the equality holds only if C coincides with $\bar{C}$. The proof is based on the construction of a certain class of admissible curves and the use of the classical Mayer field.

W. T. Reid (Albuquerque, N. M.).

Sigalov, A. G. Two-dimensional problems of the calculus of variations. Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 16–101 (1951). (Russian)

This is a detailed exposition of the fundamental results of two earlier notes [Doklady Akad. Nauk SSSR (N.S.) **70**, 769–772; **71**, 617–620 (1950); these Rev. **11**, 603, 604]. The author observes in a footnote that he had rectified independently the apparent gap in his argument to which McShane drew attention in the first cited review. Since McShane's review is very full, there is little to add here, but the follow-

ing brief comments may help to orient the reader: (1) Results similar to those of the author have been announced by others [e.g. Danskin, Bull. Amer. Math. Soc. 56, 163 (1950)] while a weaker result is automatically included in the reviewer's theory of generalized surfaces to appear in Bull. Soc. Math. France. (2) The author's line of argument makes no use of recent deep developments of the theory of area; it rests on work of McShane before 1935, on a lemma due to the reviewer (existence of a grating) and on a selection principle developed by the author, all of which concern only "Dirichlet" surfaces (or, as the author terms them, surfaces with L^2 representations). (3) The author establishes at the same time a theorem of minimum "in the small" which ensures that the diameter of a minimizing surface tends to zero with the length of its boundary. *L. C. Young.*

Mathis, H. F. An isoperimetric problem for multiple integrals in the calculus of variations. Duke Math. J. 18, 177–184 (1951).

In the present paper the author establishes the usual necessary conditions for a minimum for an isoperimetric multiple integral problem under weaker hypotheses than have been given heretofore. The normality assumptions are made. As in the case of simple integrals, the multipliers appearing in the necessary conditions are functions of the variations. *M. R. Hestenes* (Los Angeles, Calif.).

Theory of Probability

Kolmogorov, A. N. Generalization of Poisson's formula to the case of a sample from a finite set. Uspehi Matem. Nauk (N.S.) 6, no. 3(43), 133–134 (1951). (Russian)

Let an urn contain M white and $N-M$ black balls. It is stated that as $p=M/N\to 0$ and $n\to\infty$ the distribution of the number of white balls in n drawings without replacement is asymptotically equivalent to a "generalized Poisson" distribution defined by the probabilities

$$\frac{a(a-\lambda)\cdots(a-(m-1)\lambda)}{m!(1-\lambda)^m}e^{-a\omega},$$

where $\omega=-\lambda^{-1}\log(1-\lambda)$, $\lambda=n/N$. *M. Loève.*

Bergström, Harald. On asymptotic expansions of probability functions. Skand. Aktuarietidskr. 34, 1–34 (1951).

Let P and Q be distribution functions in the Euclidean space R_k. The author establishes an identity of the form

$$P^{*n}=\sum_{v=0}^{s}\binom{n}{v}Q^{*n-v}*(P-Q)^{*v}+r_n^{(s+1)}$$

where the last term can be expressed explicitly in terms of convolutions of P, Q, and $P-Q$. From this various general estimates are derived. Conditions on the bounds of the total variation $V[\varphi]$ and/or of $\sup_{x\in R_k}|\varphi(E-x)|$ (E a set in R_k) for some quantities φ involved lead to corresponding bounds for others. Thus in general the conclusions are about the orders of magnitude of the terms in the above expansion. The precise statements are too long to reproduce here. In the case of Edgeworth expansion they reduce to well-known results of Cramér and Esseen, but under different conditions of validity. The case of density functions is also included. The reviewer remarks that it would be interesting to apply these methods to obtain approximation theorems for stable distributions. *K. L. Chung* (Ithaca, N. Y.).

Kozakiewicz, W. On the necessary and sufficient conditions for the convergence of a sequence of moment generating functions. Ann. Math. Statistics 22, 478–480 (1951).

Ayant étudié dans un mémoire précédent [mêmes Ann. 18, 61–69 (1947); ces Rev. 8, 470] la continuité de la correspondance entre les fonctions de répartition à 1 dimension et leurs fonction génératrices, l'auteur envisage le même problème dans le cas à 2 dimensions (ses résultats s'étendent immédiatement à n dimensions); il prouve par exemple que: soit $\{F_n(x_1,x_2)\}$ une suite de fonctions de répartition avec

$$M_n(x_1,x_2)=\int\int_{|u_i|>\alpha_i}dF_n(u_1,u_2),\quad x_1,x_2>0,$$

$M(x_1,x_2)=$ limite supérieure de $\{M_n(x_1,x_2)\}$, $\varphi_n(t_1,t_2)=$ fonction génératrice de $F_n(x_1,x_2)$; on suppose que $\varphi_n(t_1,t_2)$ existe pour $|t_i|<\alpha_i$; la condition nécessaire et suffisante pour que $\{\varphi_n(t_1,t_2)\}$ converge pour $|t_i|<\alpha_i$ est (a) qu'il existe K tel que $M(x_1,x_2)\leqq K\exp(-|t_1|x_1-|t_2|x_2)$ pour $|t_i|<\alpha_i$, (b) qu'il existe une fonction de répartition $F(x_1,x_2)$ telle que $F(x_1,x_2)=\lim F_n(x_1,x_2)$ en tout point de continuité de $F(x_1,x_2)$. *R. Fortet* (Caen).

Darling, D. A. Sums of symmetrical random variables. Proc. Amer. Math. Soc. 2, 511–517 (1951).

Let $S_n=X_1+X_2+\cdots+X_n$ be successive sums of independent random variables with a common continuous and symmetrical distribution function. Let $p_{j,k}(n)$ be the probability that S_j is the $(k+1)$th largest among $S_0=0, S_1, S_2, \cdots, S_n$. It is proved that

$$p_{j,k}(n)=\sum_{r=\max(0,\,j+k-n)}^{\min(j,\,k)}u_r u_{k-r}u_{n-k-j+r},$$

where $u_i=C_{2i,i}/2^{2i}$. This extends a result due to E. S. Anderson [Skand. Aktuarietidskr. 32, 27–36 (1949); these Rev. 11, 256]. Two limit theorems are then given and proofs are indicated. *M. Loève* (Berkeley, Calif.).

Rosenblatt, M. On a class of Markov processes. Trans. Amer. Math. Soc. 71, 120–135 (1951).

The author relates the study of certain differential and integral equations to that of the process $(\xi(t), y(t))$ where $\xi(t)$ is a point of the product space of n independent Wiener spaces and where $y(t)=\int_0^t V(\tau,\xi(\tau))d\tau$, with various restrictions imposed on V. Among other propositions, too long to give here, he obtains extensions to the n-dimensional case of some results due to M. Kac [same Trans. 65, 1–13 (1949); these Rev. 10, 383]. *M. Loève.*

Franckx, Ed. Chaines de Markoff et échelles numériques. Trabajos Estadistica 1, 147–156 (1950). (French. Spanish summary)

The author notes a simple reciprocal relation between stationary Markov chains with r states and systems of linear recurrence equations with constant coefficients of the form $N_t=p_1N_{t-1}+\cdots+p_rN_{t-r}$ where $\sum_{i=1}^{r}p_i=1$. Bad printing and confusion between rows and columns of a matrix make the otherwise easy paper difficult to decipher.

K. L. Chung (Ithaca, N. Y.).

Kolmogorov, A. N. Solution of a problem in probability theory connected with the problem of the mechanism of stratification. Amer. Math. Soc. Translation no. 53, 8 pp. (1951).

Translated from Doklady Akad. Nauk SSSR (N.S.) 65, 793–796 (1949); these Rev. 10, 720.

Mathematical Statistics

✓ ***Linder, Arthur. Statistische Methoden für Naturwissenschafter, Mediziner und Ingenieure.** 2d ed. Verlag Birkhäuser, Basel, 1951. 238 pp. 26 Swiss francs; bound, 30 Swiss francs.

This is a revised and enlarged edition of the author's earlier book [these Rev. 7, 316]. Although the size of the book was increased by approximately fifty percent, the general character has remained unchanged. In revising the book the author was mainly concerned with the needs of the applied statistician. A rather detailed discussion of multiple and partial correlation has been added to chapters 1 and 2. Only the case of two or three independent variables is considered, this should however enable the reader to deal also with any number of independent variables. Some concepts and techniques of multivariate analysis are also presented in the new edition, such as discriminant functions and Mahalanobis' generalized distance. The most important addition to the book is the third chapter. It gives a brief discussion of the analysis of variance and thus supplies material whose absence limited the usefulness of the first edition. According to the plan of the book the first three chapters are written without mathematical proofs while chapter four gives an exposition of the mathematical theory. This chapter had to be expanded in order to supply some mathematical background for the techniques newly presented in the first three chapters of this edition. Otherwise the chapter was not changed and some details are therefore still presented in a manner objectionable to the mathematician. The progress of mathematical statistics during the last forty years has remained almost unknown in Germany and in German speaking countries. This fact adds great importance to any new publication which, like the book under review, aims at disseminating the results obtained by mathematical statisticians in the English speaking countries as well as in Scandinavia and India. It is therefore most regrettable that the mathematical part has not been more thoroughly revised. While the book will undoubtedly be helpful to applied statisticians it is still true that the needs of a mathematical reader are not yet satisfied. It might be worthwhile to note that two important methods of statistical inference are not mentioned in this book, namely sequential analysis and nonparametric tests. This reviewer hopes that they will be considered in any future edition of this book.

E. Lukacs (Washington, D. C.).

***Azorín Poh, Francisco, Ríos García, Sixto, Cansado Maceda, Enrique, y Anós y Díaz de Arcaya, Angel. Conferencias de preparacion matematica y estadistica. [Introductory Lectures in Mathematics and Statistics].** Instituto Nacional de Estadistica, Madrid, 1950. xi+186 pp.

An expository survey of the concepts of measure and integral, the Kolmogorov identification of probability with measure theory, probability distributions, and the Fisher-Neyman-Pearson theories of estimation and testing hypotheses.

D. Blackwell (Washington, D. C.).

Gumbel, E. J., and Herbach, L. H. The exact distribution of the extremal quotient. Ann. Math. Statistics **22**, 418–426 (1951).

The extremal coefficient Q is the ratio of the algebraically largest to the algebraically smallest observation in a simple sample. The authors give the cumulative distribution function of Q in terms of integrals involving the c.d.f. of the underlying distribution. There is also a detailed discussion of some special cases, particularly the case of a uniformly distributed population. *G. Elfving* (Helsingfors).

Weibull, Martin. The regression problem involving non-random variates in the case of stratified sample from normal parent populations with varying regression coefficients. Skand. Aktuarietidskr. **34**, 53–71 (1951).

Previous results of the author [same Tidskr. **33**, 137–167 (1950); these Rev. **12**, 724] are extended to more complicated situations. The usual statistics of the analysis of covariance, having chi-square, t, or F distributions if certain linear relations hold, are shown to have non-central versions of these distributions otherwise, with the usual independence properties still valid. *J. L. Hodges, Jr.* (Chicago, Ill.).

Olekiewicz, M. Determining number of independent observations n', equivalent to n observations that are not independently obtained. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **4**, 105–113 (1950). (French. Polish summary)

Suppose that b individuals are chosen at random from a population and that k observations are made on each individual with respect to a trait x. If the $n=bk$ observations thus obtained are not independent, it should be possible to take an "equivalent" sample (i.e., one containing the same amount of relevant information) by making only one observation on each of n' individuals chosen at random. The author considers several types of applications and obtains estimates and confidence limits for n' in each case, where equivalence is defined in terms of sampling variances of the statistics involved. *R. P. Peterson* (Seattle, Wash.).

Davis, R. C. Note on uniformly best unbiased estimates. Ann. Math. Statistics **22**, 440–445 (1951).

Under certain regularity hypotheses on a parametric family of distributions $F(x, \theta)$, any unbiased estimate of a function $\tau(\theta)$ whose variance attains the Bhattacharyya lower bound [Sankhya **8**, 1–14 (1946); these Rev. **8**, 524] and which is a linear combination of the first and second logarithmic derivatives of the liklihood is a function of the first logarithmic derivative only. *D. Blackwell.*

Johnson, N. L. On the comparison of estimators. Biometrika **37**, 281–287 (1950).

Two criteria for goodness of estimators, the expected squared error and Pitman's closeness criterion, are applied to several examples. Typical result: an example is studied in which the maximum likelihood estimator has asymptotically twice the expected squared error of another estimator. Charts are given for applying the Pitman criterion to biased correlated normal estimators. *J. L. Hodges, Jr.*

Neyman, J., and Scott, Elizabeth L. On certain methods of estimating the linear structural relation. Ann. Math. Statistics **22**, 352–361 (1951).

The methods of A. Wald [Ann. Math. Statistics **11**, 285–300 (1940); these Rev. **2**, 108] and J. Hemelrijk [Nederl. Akad. Wetensch., Proc. **52**, 995–1005 = Indagationes Math. **11**, 374–384 (1949); these Rev. **11**, 529] for the statistical treatment of an "error model" are subjected to a penetrating discussion. It turns out that if no specific information is available about the exact relationship that is assumed to subsist between the error-free variables, the methods will be either biased or non-consistent in Fisher's sense.

H. Wold (Uppsala).

Sandelius, Martin. On non-sequential estimation when the sample size is a random variable. Kungl. Lantbrukshögskolans Annaler **17**, 400–406 (1950).
Expository paper.

Blackwell, David. On the translation parameter problem for discrete variables. Ann. Math. Statistics **22**, 393–399 (1951).

Suppose the parameter h is to be estimated from the observation of $y=(x_1+h, \cdots, x_N+h)$, the loss depending only on the difference between h and the estimate, and the distribution of $(x_1, \cdots, x_N)$ being known. Following the suggestion of Pitman [Biometrika **30**, 391–421 (1939)], Girshick and Savage [Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 53–73, University of California Press, Berkeley and Los Angeles, 1951] have shown that under certain conditions there exists a minimax estimate of the form $\bar{x}+v(x_1-\bar{x}, \cdots, x_N-\bar{x})$. The paper under review shows that minimax estimates of this form are admissible, in the technical sense, under rather general conditions and gives two examples illustrative of conditions in which they are not admissible. *L. J. Savage* (Paris).

Dresselaers, Céline, et Gillis, Paul P. Tests de signification pour hypothèses composées unilatérales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 449–458 (1951).

Robertson, A. The analysis of heterogeneity in the binomial distribution. Ann. Eugenics **16**, 1–15 (1951).

Let p_k be a finite sequence of identically distributed independent random variables confined to the interval $(0, 1)$ with mean π and variance σ^2, n_k a corresponding sequence of integers, and x_k a corresponding sequence of independent random variables binomially distributed with the parameters p_k and n_k. This paper extensively considers, with special reference to genetics, two problems of estimation based on observation of the x_k and n_k: First, estimation of σ^2 when $\sigma^2 \ll \pi(1-\pi)$. Second, estimation of the one unknown parameter when the distribution of the p_k is confined to two stated values. New methods are suggested for each of these problems and compared with old ones. *L. J. Savage.*

Smith, C. A. B. A test for heterogeneity of proportions. Ann. Eugenics **16**, 16–25 (1951).

This paper is closely related to that of Robertson reviewed above, extending the work on binomial distributions there to multinomial distributions. Author's summary: "A new criterion for testing for heterogeneity in a contingency table is proposed. Reasons are given for supposing that this gives a more powerful test for χ^2, and also that, unlike χ^2, it can be used when the numbers in the cells are small. A simple approximate method of using the test is explained. When heterogeneity is present, this method will give unbiased estimates of the variances and covariances of the probabilities." *L. J. Savage* (Paris).

Barnard, G. A. On the Fisher-Behrens test. Biometrika **37**, 203–207 (1950).

It is remarked that Stein's double sampling procedure [Ann. Math. Statistics **16**, 243–258 (1945); these Rev. **7**, 213] can be extended to give a test for the equality of means of two normal populations [cf. the paper reviewed below]. This test is similar to the Fisher-Behrens test, but also is valid according to the Neyman-Pearson theory. Hence it is suggested that if any pair of samples is considered as having arisen from a double sampling procedure, the objection to the Fisher-Behrens test (that the probability of rejecting the hypothesis when true is not necessarily equal to the chosen significance level α) is no longer valid. *D. G. Chapman.*

Chapman, Douglas G. Some two sample tests. Ann. Math. Statistics **21**, 601–606 (1950).

Stein [same Ann. **16**, 243–258 (1945); these Rev. **7**, 213] showed how a two stage sequential plan for sampling from a normal population of unknown variance, could produce a t-statistic centered on the population mean. The present paper employs this device to give a solution to the Behrens-Fisher problem, in a generalized form, and to a related regression problem. [The same solution was reached independently by Ruben, quoted by Barnard in the paper reviewed above.] *J. L. Hodges, Jr.* (Chicago, Ill.).

Marshall, Andrew W. A large-sample test of the hypothesis that one of two random variables is stochastically larger than the other. J. Amer. Statist. Assoc. **46**, 366–374 (1951).

Let x, y have the continuous cumulative distribution functions $F(a)$ and $G(a)$ respectively. Let $F_n(a)$, $G_m(a)$ be the sample distributions of a sample of n of the x's and m of the y's respectively. It is desired to test the hypothesis $G(x)=F(x)$ against the alternative that x is stochastically larger than y, that is to say $F(a) \leqq G(a)$ for all a and $F(b)<G(b)$ for at least one b. The author observes that Wilcoxon's test [Biometrics **1**, 80–83 (1945)], studied in detail by the reviewer and D. R. Whitney [Ann. Math. Statistics **18**, 50–60 (1947); these Rev. **9**, 151], is too cumbersome for the treatment of large samples. He first remarks that in this case Max $(G_m(a)-F_n(a))$ could be used as a test statistic, since its limit distribution is known from a theorem of Smirnov [Bull. Math. Univ. Moscou **2**, no. 2 (1939); these Rev. **1**, 345]. However, in the case of grouped data the test is not directly applicable. The author therefore proposes the following test. Let a_i, $i=1, \cdots, j$ be j predetermined points. Put $z_i=G_m(a_i)-F_n(a_i)$ and $S=\sum_{i=1}^{j} z_i$. The critical region is to be $S \geqq K$. The author derives the mean and variance of S and shows that S is in the limit normally distributed. The asymptotic power efficiency of S is stated to be .64 for $j=1$, .91 for $j=5$ and .94 for $j=10$. The author's investigations on the efficiency of the test will be reported in a later paper. *H. B. Mann* (Columbus, Ohio).

Fraser, D. A. S. Sequentially determined statistically equivalent blocks. Ann. Math. Statistics **22**, 372–381 (1951).

The author presents an extension of the Wilks-Wald-Tukey method for obtaining tolerance regions from a sample by cutting off statistically equivalent blocks from the space of the random variables sampled [cf., e.g., Tukey, Ann. Math. Statistics **18**, 529–539 (1947); these Rev. **9**, 295]. Earlier, a fixed sequence of functions of the random variables was used for this purpose. Attention is here drawn to the fact that the functions used for cutting off successive blocks may depend on the values of the previous functions. In this way it is possible to meet requirements on the shape of the tolerance region more closely. The distribution of the probability coverages remains the same as before. The discontinuous case requires a special treatment.
G. Elfving (Helsingfors).

Fieller, E. C., and Smith, C. A. B. Note on the analysis of variance and intraclass correlation. Ann. Eugenics **16**, 97–104 (1951).

The authors discuss the relationship between the analysis of variance and intraclass correlation. They show that in certain genetical applications, e.g. sib-sib correlation, the approach to intraclass correlation by separating the variance into components has advantages over the classical method. The intraclass correlation is $\rho=v_b/(v_b+v_w)$, where v_w is the average variance within classes (sibships) and v_b is the variance between the true means of the classes. Unbiased estimates for v_b and v_w are obtained. *R. P. Peterson.*

Wold, Herman. On least square regression with autocorrelated variables and residuals. Bull. Inst. Internat. Statist. **32**, no. 2, 277–289 (1950).

The classical least squares regression procedure is extended to time series where the variables and residuals may be inter- and auto-correlated. The extended results are proved by methods similar to the classical theorem. Formulae are given for the asymptotic variances and covariances of the estimated regression coefficients for the case of strictly linear regression and for curvilinear regression where linear regression is regarded as a graduation. (The author has pointed out subsequently that these formulae are not valid in this second case.) The results are used to derive a large sample test for the regression coefficients, which is again similar to that for the usual situation. Finally a related study by Cochrane and Orcutt [J. Amer. Statist. Assoc. **44**, 32–61 (1949)] is discussed and certain defects of this study are pointed out. *D. G. Chapman.*

Halperin, Max. Normal regression theory in the presence of intra-class correlation. Ann. Math. Statistics **22**, 573–580 (1951).

For strictly linear regression in the case of intercorrelated disturbances, the reviewer has shown that the least-squares regression coefficients are unbiased and given large-sample formulae for their standard error [see the preceding review]. Dealing with the small-sample aspect of the same situation the author shows that the classical tests of significance are valid in the special case when all intercorrelations are equal. *H. Wold* (Uppsala).

Quenouille, Maurice H. Trend elimination in time-series. Bull. Inst. Internat. Statist. **32**, no. 2, 290–299 (1950).

The author concerns himself with the effect of a trend elimination upon a subsequent estimation of the stationary component. His results indicate that the moving average method of trend elimination both distorts the correlogram of the stationary component, and renders it less sensitive to changes in the parameters, this implying a considerable loss in the precision of the parameter estimates. When the trend is eliminated by curve-fitting, then the correlogram estimates are still biassed. However, as the author has shown earlier, this bias may be removed as far as terms of order less than n^{-2} if a correlation coefficient is estimated by $R=2r-\frac{1}{2}(r_1+r_2)$, where r, r_1 and r_2 are the usual estimates for the whole series and its two halves. This correction is tested here on artificial material, and shows itself to be an effective one.

P. Whittle (Uppsala).

Mathematical Economics

Robinson, Julia. An iterative method of solving a game. Ann. of Math. (2) **54**, 296–301 (1951).

The author establishes the convergence of an iterative process suggested by the reviewer for approximating solutions for each player in a finite two-person zero-sum game. The method corresponds to each player choosing in turn his best pure strategy against the accumulated mixed strategy of his opponent up to then. The proof, an induction on the size of the matrix, rests upon the appropriate definition of "vector systems", of which the suggested iteration is a special case. *G. W. Brown* (Santa Monica, Calif.).

✓★**Brown, George W. Notes on the solution of linear systems involving inequalities.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 137–140. Harvard University Press, Cambridge, Mass., 1951. $8.00.

A simple iterative process is proposed for numerically calculating optimal mixed strategies for a finite two-person zero-sum game: Each player in turn chooses one of his pure strategies which is best against the mixture constituted by his opponent's plays to date. At each iterative step the amount of calculation is directly proportional to the linear dimensions of the pay-off matrix. Relations are shown between the game-theory problem and the minimization of a linear form subject to linear inequalities. There is a numerical example with a 3×4 pay-off matrix. Reviewer's note: J. Robinson [see the preceding review] has proved the convergence of Brown's process. *G. E. Forsythe.*

Nash, John. Non-cooperative games. Ann. of Math. (2) **54**, 286–295 (1951).

The general finite n-person game has previously been analyzed by considering the related two-person games obtained by partitioning the set of players into two disjoint subsets or coalitions [von Neumann and Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944; these Rev. **6**, 235]. This paper gives an entirely new treatment of such games in which cooperation (coalitions) are excluded. The main result is a generalization of the fundamental theorem of two-person zero-sum games (existence of a value) to general n-person games, and may be described as follows: Let $s_1, s_2, \cdots, s_n$ denote mixed strategies for players $1, 2, \cdots, n$, and let $p_i(s_1, s_2, \cdots, s_n)$ be the pay-off function to the ith player, where, as usual, each p_i is an n-linear function of the s_i's. An n-tuple of strategies $(s_1, s_2, \cdots, s_n)$ is called an "equilibrium point" if, for all i, $p_i(s_1, s_2, \cdots, s_n) \geqq p_i(s_1, s_2, \cdots, s_i', \cdots, s_n)$, where s_i' is any strategy of the ith player. Theorem: Every finite n-person game has an equilibrium point. Making use of this concept the author defines the notion of solution of a game (which, however, need not exist). Further sections are devoted to analyzing the structure of solutions and giving examples. *D. Gale* (Providence, R. I.).

✓★**Wood, Marshall K., and Dantzig, George B. The programming of interdependent activities: General discussion.** Activity Analysis of Production and Allocation, pp. 15–18. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

"This . . . was originally published, in a more extensive form, in Econometrica, Vol. 17, July-October, 1949, pp. 193–199 [these Rev. 11, 193]" (authors' footnote).

K. J. Arrow (Stanford University, Calif.).

✓ ***Dantzig, George B. The programming of interdependent activities: Mathematical model.** Activity Analysis of Production and Allocation, pp. 19–32. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

"The present paper represents a revision and extension of an earlier paper which appeared in Econometrica, Vol. 17, July-October, 1949, pp. 200–211 [these Rev. 11, 193]" (author's footnote). Apart from somewhat greater detail, the principal novelty is the introduction of a postulate system for a continuous technology in which the time translation of any activity is itself a new activity.

K. J. Arrow (Stanford University, Calif.).

✓ ***Smith, Harlan M. Uses of Leontief's open input-output models.** Activity Analysis of Production and Allocation, pp. 132–141. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The author points out that a Leontief open input-output model cannot answer the question, what is the effect of a given autonomous increase in some final demands. Rather it tells what is the industrial structure associated with a given set of final demands, autonomous and induced. The author suggests using a closed model with autonomous expenditures as such separated out as a final bill of goods; however, the statistically estimated input-output coefficients would have to be revised to include only induced expenditures. *K. J. Arrow.*

Samuelson, Paul A. The problem of integrability in utility theory. Economica N.S. 17, 355–385 (1950).

The author reviews the history of the integrability controversy in the utility theory of consumers' demand. The question is that of the economic meaning of the conditions imposed on demand functions by the requirement that they be derived from the principle of maximizing some utility function under a budget restraint. The various stages in the development of the subject from the original formulations of Antonelli, Fisher, and Pareto to the author's own "revealed preference" approach and the recent important contribution of Houthakker [Economica N.S. 17, 159–174 (1950); these Rev. 13, 146] are summarized. It is then shown that most of the arguments used to support the admissibility of the non-integrable case are not defensible. In a mathematical appendix, the conditions on demand functions implied by utility maximization are stated and related to a system of differential equations. A proof that the integrability conditions of the latter are necessary and sufficient for the existence of a utility function is sketched, as is Georgescu-Roegen's discussion of the various non-integrable cases in three-dimensional space. *K. J. Arrow.*

✓ ***Georgescu-Roegen, Nicholas. The aggregate linear production function and its applications to von Neumann's economic model.** Activity Analysis of Production and Allocation, pp. 98–115. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

The author generalizes somewhat the production system studied by J. von Neumann [Erg. Math. Kolloq. 8, 73–83 (1937)]. A transformation may be denoted by

$$\begin{pmatrix} a_1, a_2, \cdots, a_n \\ b_1, b_2, \cdots, b_n \end{pmatrix},$$

where the a_k's denote inputs and the b_k's outputs. It is assumed (a) that multiplying all components by a positive constant leads to a new transformation; (b) the sum of two transformations is a transformation; (c) $a_k>0$ for at least one k; (d) $a_k+b_k>0$ for all k. The set of transformations forms a convex cone in $2n$-space; the cone is assumed closed. If $y_1, \cdots, y_n$ are the prices of the various commodities (assumed non-negative), then $\phi=(\sum b_k y_k)/(\sum a_k y_k)$ is the "return to the dollar" for the process of transformation; that process is to be chosen which maximizes ϕ, while competition causes prices to vary so as to minimize ϕ for the chosen process. Von Neumann's proof that such a process and set of prices exist is considerably simplified and extended to the case where the cone of transformations need not be polyhedral. The equilibrium value of ϕ may be termed the equilibrium rate of growth M. It is shown that for any commodity $b_k/a_k \geqq M$ in the equilibrium process; further, for any commodity for which the strict inequality holds, the price $y_k=0$. In other words, there is a set of commodities whose output increases geometrically at the rate M; all commodities whose rate of output increases more rapidly must be free. The equilibrium rate of growth is unique, but neither the equilibrium process nor the equilibrium prices need be. *K. J. Arrow* (Stanford University, Calif.).

Morehouse, N. F., Strotz, R. H., and Horwitz, S. J. An electro-analog method for investigating problems in economic dynamics: Inventory oscillations. Econometrica 18, 313–328 (1950).

This paper discusses the use of the Aeracom-type analog-computer in the solution of dynamic economic models. An electrical analog is constructed for a particular economic model, and the mathematical solution of the electrical system is taken as the solution of the economic model. The method is illustrated by a simple industry inventory model. There is postulated a linear demand function (1) $P_d=\alpha_1-\beta_1 Q_s$, and a linear output function (2) $P_s=\alpha_2+\beta_2 Q_p$, where Q_s is quantity exchanged, P_d is demand price for Q_s, Q_p is quantity produced, and P_s is the minimum price at which Q_p will be produced. Q_p-Q_s is the inventory increment. The industry will seek a certain level of inventories, Q_i.

In equilibrium, $P_d=P_s=P$ (market price), $Q_s=Q_p$, and $Q_i=$constant. From an initial equilibrium position (P^0, Q^0) disequilibrium is introduced by a rise in α_1 at T^0. To explain the time path to the new equilibrium (P^1, Q^1) at T^1, two additional functions, whose nature depends on the market properties postulated, are required. Here two linear time lags are assumed. The industry responds to the shift in demand by both a gradual reduction in Q_i and by a gradual increase in production to rebuild Q_i and to meet increased sales. Since the industry may overshoot the new equilibrium level, oscillations are introduced. The behavior equations assumed here are:

$$P_d-P_s=\lambda_1\dot{Q}_s+\lambda_2\dot{Q}_p, \tag{3}$$

and

$$P_d-P^0=\lambda_1\dot{Q}_s+\gamma^{-1}\int_{T^0}^{T}(\Delta Q)dT+P_i, \tag{4}$$

where $\Delta Q=Q_s-Q_p$, T is time, P_i is an inventory control function, and the dot signifies a derivative with respect to time. The quantities λ_1 and λ_2 are inertia elements reflecting response lags in increasing sales and increasing output respectively. In (4), the inventory condition is given by $\gamma^{-1}\int_{T^0}^{T}(\Delta Q)dT+P_i$, where γ^{-1} represents the degree of in-

ventory inflexibility. P_i controls the magnitude of and the method of approach to the new equilibrium level of Q_i. It is assumed here that the change in the equilibrium position causes a shift in the "reservation demand" function of the firm and that this shift occurs exponentially and in such a manner that $Q_i^0 = Q_i^1$ for a sufficiently large value of T. This requirement is satisfied by putting

$$P_i = (P^1 - P^0)(1 - \epsilon^{-\rho(T-T^0)}).$$

The corresponding electrical circuit equations may be written

$$E_1 = V_1 - R_1 I_1, \tag{5}$$

$$E_2 = V_2 + R_2 I_2, \tag{6}$$

$$E_1 - E_2 = L_1 \dot{I}_1 + L_2 \dot{I}_2, \tag{7}$$

$$E_1 - E^0 = L_1 \dot{I}_1 + \frac{1}{C}\int_\rho^t (I_1 - I_2)dt + E_i, \tag{8}$$

where $E_i = (E^1 - E^0)(1 - \epsilon^{-(R_2/L_2)(t-\rho)})$, R is resistance, L is inductance, C is capacitance, V is battery voltage across one or more circuit elements, and I is current. From these equations and Kirchoff's laws an electrical circuit was devised. This circuit was plugged into the Aeracom and the values of the elements set to simulate an initial equilibrium position of the model. The equilibrium was disrupted by increasing the value of V_1, and the behavior of selected variables observed as time series on the oscilloscope screen. The solutions for a variety of new conditions were obtained and photographed. There follows a discussion of the potential applicability of the electro-analog method to more complex economic models. *M. P. Stoltz* (Providence, R. I.).

Mathematical Biology

Malécot, G. Quelques schemas probabilistes sur la variabilité des populations naturelles. Ann. Univ. Lyon. Sect. A. (3) **13**, 37–60 (1950).

The main part of the paper centers around the probability that two homologous autosome loci of a single organism taken at random in the nth generation be the same as those of an ancestor. Mutation is taken into account. The effect of migration is studied on the basis of the probability that a person born in one group of population have their parents born in another group. Various expressions of this probability reflecting geographic factors are considered. Methods for a statistical verification of the theory are indicated.

I. Opatowski (Chicago, Ill.).

Skellam, J. G. Random dispersal in theoretical populations. Biometrika **38**, 196–218 (1951).

L'auteur étudie le problème écologique de peuplement d'un biotrope par une espèce animale ou végétale et de formation et d'évolution d'une population. En prenant comme point de départ la notion statistique des déplacements arbitraires, il étudie l'expansion d'une population à partir d'un centre et en trouve la vitesse constante en bon accord avec les observations (chênes; Ondatra zibethica). Une analogie avec la théorie de diffusion permet de compléter l'équation logistique de Verhulst par un terme exprimant la diffusion, ce qui donne l'équation:

$$\partial\psi/\partial t = e\psi - h\psi^2 + \sigma(\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2)$$

en désignant par $\psi(x, y, t)$ la densité de population et en supposant la densité initiale connue. Le problème mathématique est difficile et l'auteur étudie soit des cas particuliers (distribution linéaire; population malthusienne), soit un algorithme d'approximations successives appliqué numériquement à l'état stationnaire, sans justification mathématique, exclusivement par la voie numérique. Dans le cas des populations clairsemées, la méthode analytique n'est plus valable, et l'auteur utilise la méthode statistique et la loi de distribution de Poisson pour obtenir une équation récurrente dont la solution approchée nous ramène à la loi logistique. Quelques considérations biologiques et la bibliographie terminent le mémoire. *V. A. Kostitzin.*

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. VII. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 286–290 (1951).

The author extends his variational expression of the laws of genetics to the inheritance of sex and an expression of the so-called genetic "Principle of Finality".

A. S. Householder (Oak Ridge, Tenn.).

Rapoport, Anatol. "Addition" and "multiplication" theorems for the inputs of two neurons converging on a third. Bull. Math. Biophys. **13**, 179–188 (1951).

The output of a neuron innervated by two other neurons which, in turn, are subjected to two independent Poisson showers of stimuli, is derived as a function of the frequencies of the Poisson showers under two distinct assumptions, (1) where either of the two neurons can fire the third, and (2) where the stimuli from both neurons must impinge within a certain time interval to fire the third. For very small frequencies, the output of the third neuron is very nearly the sum of the input frequencies in the first case and proportional to the product of the input frequencies in the second case. (From the author's summary.)

A. S. Householder (Oak Ridge, Tenn.).

Opatowski, I. On simultaneous determinations of the permeability of a membrane and of the diffusion coefficient in an adjacent medium. Bull. Math. Biophys. **13**, 119–132 (1951).

L'auteur réduit le problème de détermination de la perméabilité d'une membrane et du coefficient de diffusion d'un fluide dans un récipient adjacent à la résolution d'une équation aux dérivées partielles assez simple, ceci en sacrifiant ce qui différentie une membrane d'un milieu fluide idéalisé. En supposant la membrane limitée par deux plans parallèles ($x = -a$, $x = 0$), le récipient ($0 < x < c$) à concentration initiale nulle et le demi-espace ($x < -a$) rempli par un gaz à concentration constante C, l'auteur représente le processus de diffusion par les équations $\partial C_a/\partial t = D_a \partial^2 C_a/\partial x^2$, $\partial C_b/\partial t = D_b \partial^2 C_b/\partial x^2$, en désignant par $C_a(x, t)$, $C_b(x, t)$ les concentrations resp. dans la membrane et dans le récipient et par D_a et D_b les coefficients de diffusion correspondants. Ces fonctions sont soumises aux conditions suivantes

$$D_a \left.\frac{\partial C_a}{\partial x}\right|_{x=0} = D_b \left.\frac{\partial C_b}{\partial x}\right|_{x=0}, \quad \left.\frac{\partial C_b}{\partial x}\right|_{x=b} = 0,$$

$$C_a(-a, t) = C, \quad C_a(0, t) = C_b(0, t).$$

La transformation de Laplace permet de résoudre ces équations. Les solutions n'étant pas maniable, l'auteur discute quelques formules approchées. Seul l'expériment nous dira si les hypothèses simplificatrice de l'auteur (absence d'adsorption, réductions à une dimension, etc.) donnent effectivement une représentation convenable du processus étudié. *V. A. Kostitzin* (Paris).

TOPOLOGY

Willcocks, T. H. A note on some perfect squared squares. Canadian J. Math. 3, 304–308 (1951).

This paper gives methods for dissecting a square into a finite number of unequal non-overlapping squares, in which the sides of the square pieces can be expressed by relatively small integers [cf. W. T. Tutte, same J. 2, 197–209 (1950); these Rev. 12, 118]. The basic step is the formation of two "equivalent" rectangles, i.e., of two different dissections of the same rectangle. Methods are given for "bordering" suitable pairs of equivalent rectangles to obtain others. The author further notes that one of the two rectangle dissections involved may be allowed to have two pieces equal, and in this way obtains the simplest "perfect square" at present known, consisting of a square 175×175 dissected into 24 unequal integral square pieces. *A. H. Stone.*

†Aleksandrov, P. S., and Uryson, P. S. On compact topological spaces. Trudy Mat. Inst. Steklov. 31, 95 pp. (1950). (Russian)

This is a revised second edition (in Russian, however) of the celebrated, but not easily accessible, article by Urysohn and Alexandroff, "Mémoire sur les espaces topologiques compactes" [Verh. Kon. Akad. Wetensch. Amsterdam. Afd. Natuurk. Sect. 1. 14, no. 1 (1929)]. There is a foreword by Alexandroff, and each of the five chapters has a concluding page of comments by him. The revision is said to be entirely in the nature of a reexposition, following very closely on the lines of the older paper. It is of interest that the material of this remarkable paper was the work of the year 1922, the results being announced in the next few years in the Bulletin International de l'Académie Polonaise des Sciences et des Lettres and the Mathematische Annalen. *L. Zippin* (Brooklyn, N. Y.).

Dowker, C. H. On countably paracompact spaces. Canadian J. Math. 3, 219–224 (1951).

A topological space X is "countably paracompact" if every covering of X by countably many open sets has a locally (neighbourhood) finite refinement. Theorem 1. The product $X\times Y$ of a countably paracompact space X and a compact space Y is countably paracompact. Theorem 2. A normal space X is countably paracompact if and only if every countable covering of X has a point-finite refinement; other related equivalent conditions are also given, leading to the corollary that every perfectly normal space is countably paracompact. Finally it is shown that the following properties of a topological space X are equivalent: (1) X is countably paracompact and normal, (2) $X\times I$ is normal, where I is the unit interval $\{0\leqq x\leqq1\}$, (3) $X\times Y$ is normal whenever Y is a compact metric space, (4) if g and h are, respectively, lower and upper semicontinuous real functions on X, and $h(x)<g(x)$ (all $x\in X$), there exists a continuous real function f such that $h(x)<f(x)<g(x)$ for all $x\in X$. (The equivalence of (2) and (4) was conjectured by Eilenberg.) It is an open question whether every normal Hausdorff space is countably paracompact. *A. H. Stone.*

Arens, Richard, and Dugundji, James. Topologies for function spaces. Pacific J. Math. 1, 5–31 (1951).

Let X, Y, Z be topological spaces, let Z^Y be the space of continuous mappings of Y into Z. Each mapping $g: X\times Y\to Z$ defines an associated mapping $g^*: X\to Z^Y$ by $g^*(x)(y)=g(x,y)$ and conversely. A topology on Z^Y is called "proper" if for every X the continuity of g implies the continuity of g^*, "admissible" if for every X the continuity of g^* implies the continuity of g. Admissibility is equivalent to the continuity of ω: $Y\times Z^Y\to Z$ defined by $\omega(y,f)=f(y)$ for all $f\in Z^Y$. An exhaustive study is undertaken of the position of proper and admissible topologies in the lattice L of all topologies on Z^Y. The following are some of the results, negative results always being substantiated by examples. Every admissible topology is at least as fine as every proper topology; a topology finer than an admissible topology is admissible, a topology coarser than a proper topology is proper. The proper topologies form an ideal in L with a greatest (finest) element. There is not always a coarsest admissible topology; for example, if Y is completely regular but not locally compact, and if Z is the real line, there is no coarsest admissible topology, although this is not necessarily true if "completely regular" is replaced by "Hausdorff". A proper admissible topology, if it exists, is unique. For any Y, the compact-open topology is proper, but it is not always admissible and not even always the finest proper topology; however, for a metric Z containing a non-degenerate arc, and a completely regular Y, it is the finest proper set-open topology, while for a locally compact regular Y it is admissible. If X is an arbitrary set, and Y a locally compact regular space, necessary and sufficient conditions are given that a topology T in $X\times Y$ be a product topology with one factor the space Y. *S. B. Myers* (Ann Arbor, Mich.).

Nagata, Jun-iti. On a necessary and sufficient condition of metrizability. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 93–100 (1950).

The principal theorem is that a regular topological space R is metrizable if and only if there exists a sequence of locally (neighborhood) finite open coverings $\mathfrak{U}_n$ ($n=1, 2, \cdots$) of R whose sets U_n together form a basis for the open sets of R. [This is closely related to results obtained independently by R. H. Bing; see the following review.] An equivalent condition is given in terms of continuous real functions on R, and a similar criterion is given for R to be completely metrizable: it is necessary and sufficient that R be fully normal and topologically complete [E. Čech, Ann. of Math. 38, 823–844 (1937)] and have a sequence of locally finite open coverings $\mathfrak{U}_n$ such that, given any distinct points a, $b\in R$, there exists some $U_{n\beta}$ with $a\in U_{n\beta}$ and b non-$\in\bar{U}_{n\beta}$. It is also deduced that a T_1 space which has a locally finite covering by closed metrizable sets is itself metrizable. *A. H. Stone* (Manchester).

Bing, R. H. Metrization of topological spaces. Canadian J. Math. 3, 175–186 (1951).

A collection $\mathfrak{H}$ of sets H_α ($\alpha\in A$) is called "discrete" if (1) the closures $\bar{H}_\alpha$ are disjoint, (2) every union of the form $\bigcup\{\bar{H}_\beta \mid \beta\in B, B\subset A\}$ is closed. A regular topological space R is shown to be metrizable if and only if it is "perfectly screenable", i.e., there exists a sequence $\mathfrak{H}_n$ ($n=1, 2, \cdots$) of discrete collections of open sets $H_{n\alpha}$ which together form a basis for the open sets of R. An equivalent condition (suggested by E. E. Floyd, and slightly reworded in this review) is that there exists a sequence $\mathfrak{H}_n$ of open coverings of R, satisfying (2) for each n, such that for each $p\in R$ the stars $\mathrm{St}(p, \mathfrak{H}_n)$ form a basis for the neighborhoods of p. [Cf. the paper by Nagata reviewed above.] Besides giving other conditions for metrizability, the author analyses the impli-

cations between several related properties; we mention some representative results. A space is "strongly screenable" if each open covering can be refined by one which can be decomposed into a sequence of discrete collections; it is "collectionwise normal" if, for each discrete collection $\mathfrak{X}=\{X_\alpha\}$ of sets, $\bigcup X_\alpha$ can be covered by a disjoint collection of open sets each meeting at most one set X_α. A perfectly screenable space is strongly screenable; a regular strongly screenable space is collectionwise normal, but need not satisfy the first axiom of countability. The class of "developable" spaces, which includes Moore spaces, is studied in further detail; for example, a collectionwise normal Moore space is metrizable. Full normality implies collectionwise normality, which in turn implies normality; the converses are false. The paper includes some illuminating counterexamples, e.g. (a) a non-normal Moore space S with an open covering $\mathfrak{H}$ such that each point $p\in S$ belongs to only finitely many sets of $\mathfrak{H}$ and each star $\mathrm{St}(p, \mathfrak{H})$ is metrizable, (b) a perfectly normal space which is not collectionwise normal. *A. H. Stone* (Manchester).

Bing, R. H. Snake-like continua. Duke Math. J. 18, 653–663 (1951).

An ϵ-covering of a set M is a finite covering C of M by open sets of diameter less than ϵ. Let $G(C)$ be the abstract graph with the elements of C as vertices, two vertices being joined by a unique edge if and only if they intersect. If $G(C)$ is a broken line (or a tree) then C is an ϵ-chain (or an ϵ-tree-chain). A continuum is a compact metric connected space. A continuum M is snake-like (or tree-like) if for every $\epsilon>0$ it can be covered by an ϵ-chain (or an ϵ-tree-chain). The author gives an account of the known results on snake-like continua, together with new results of which the following are typical. (1) Every snake-like continuum has an uncountable collection of disjoint homeomorphic images in the plane. (2) Each plane continuum which does not contain a continuum which separates the plane is tree-like. (3) If S is an atriodic continuum, and every subcontinuum of S is unicoherent and decomposable, then S is snake-like. (A continuum M is a triod if there is a subcontinuum N of M such that $M-N$ is the sum of three mutually separated sets; S is atriodic if S contains no triod. M is decomposable if M is the sum of two proper subcontinua of M.) (4) If the snake-like continuum M is irreducible between two of its end-points p and q, then for every $\epsilon>0$ there is an ϵ-chain C, covering M, such that p and q lie in the end-points of $G(C)$. (The point p is an end-point of M if for every $\epsilon>0$ there is an ϵ-chain C, covering M, such that p lies in an end-point of $G(C)$.) *E. E. Moise.*

Bing, R. H. Concerning hereditarily indecomposable continua. Pacific J. Math. 1, 43–51 (1951).

The terminology is that of the previous review. A continuum is hereditarily indecomposable if it contains no decomposable continuum. (I) Any two nondegenerate snake-like hereditarily indecomposable continua M and M' are homeomorphic; and if M and M' are irreducible, from p to q and from p' to q' respectively, then the homeomorphism can be chosen so as to throw p onto p' and q onto q'. Such a continuum is thus homeomorphic to a pseudo-arc, defined by the reviewer [Trans. Amer. Math. Soc. 63, 581–594 (1948); these Rev. 10, 56] and previously described, differently, by Knaster [Fund. Math. 3, 247–286 (1922)]. (II) In any Euclidean space, or Hilbert space, almost all continua are pseudo-arcs; that is, if the space of continua is topologized by the Hausdorff distance, the pseudo-arcs form a dense G_δ. (III) The image of a snake-like continuum (or a pseudo-arc) under a non-trivial monotone transformation is a snake-like continuum (or a pseudo-arc). (IV) There are as many topologically different hereditarily indecomposable plane continua as there are real numbers. *E. E. Moise* (Ann Arbor, Mich.).

Bing, R. H. Higher-dimensional hereditarily indecomposable continua. Trans. Amer. Math. Soc. 71, 267–273 (1951).

Knaster [Fund. Math. 3, 246–286 (1922)] gave an example of a continuum which is hereditarily indecomposable, i.e., no subcontinuum is the union of two proper sub-subcontinua. Knaster's continuum is one-dimensional, and it has been an open question as to whether all such continua are one-dimensional; indeed, until recently [Bing, see the preceding review] it has seemed possible that all such continua be homeomorphic to the one constructed by Knaster. In this paper the author shows that there are hereditarily indecomposable continua of all dimensions.

The idea of the proof is the following. Call an arc e-crooked if for each pair of points p, q of the arc there are points r, s of the arc in order p, q, r, s such that $\mathrm{dist}(p, s)<e$ and $\mathrm{dist}(r, q)<e$. The intersection of a monotone decreasing sequence S_i of locally connected continua, such that each arc in S_i is $1/i$-crooked, is heritarily indecomposable. If each S_i separates Euclidean n-space (or the Hilbert cube) between two points, so also does the intersection, which must then be of dimension at least $n-1$ (respectively, infinite).

There are amusing corollaries. For example: If A and B are closed disjoint subsets of a continuum C there is a closed subset D separating them such that each component of D is hereditarily indecomposable. Two distinct points p, q of Euclidean n-space may be separated by a compact set C such that each continuum containing p and q contains a component of C. *J. L. Kelley* (New Orleans, La.).

Bing, R. H., and Floyd, E. E. Coverings with connected intersections. Trans. Amer. Math. Soc. 69, 387–391 (1950).

If G is a collection of subsets of a set, then a subintersection of G is a non-null set which is the common part of the elements of a subcollection of G. Let X denote a compact, locally connected, metric continuum. The author shows first that X has a countable basis whose subintersections are connected and uniformly locally connected. He next establishes that X has such a basis with the additional property that if G^1 is a subcollection of G, then the intersection of the closures of the elements of G^1 is the closure of the intersection of the elements of G^1. In consequence, for each positive integer i, X is the sum of a finite $(1/i)$-collection G_i of continuous curves such that each subintersection of $\sum G_i$ is a continuous curve. *W. W. S. Claytor* (Washington, D. C.).

Moise, Edwin E. A note of correction. Proc. Amer. Math. Soc. 2, 838 (1951).

The author states that the proof of Theorem 1 of an earlier paper [Bull. Amer. Math. Soc. 55, 1111–1121 (1949); these Rev. 11, 734] is erroneous, but that Theorem 2 is still true.

Mickle, Earl J. Fréchet and Kerékjártó equivalence. Proc. Amer. Math. Soc. 2, 518–521 (1951).

Let P_1, P_2 be homeomorphic Peano spaces, let $\mathcal{P}$ be a metric space with distance $d(p, p')$, p, $p'\in\mathcal{P}$, let T_i: $P_i\to\mathcal{P}$,

$i=1, 2$, be any two continuous mappings from P_1 and P_2 into $\mathcal{P}$. The mappings T_1 and T_2 are said to be Fréchet-equivalent (F) if for any $\epsilon>0$ there is a homeomorphism h_ϵ from P_1 onto P_2 such that $d[T_1(p), T_2(h_\epsilon p)]<\epsilon$ for any point $p \in P_1$. T_1 and T_2 are said to be Kerékjártó-equivalent (K) if T_1, T_2 admit of monotone-light factorizations $T_1=lm_1$, $T_2=lm_2$, $m_1: P_1 \to \mathfrak{M}$, $m_2: P_2 \to \mathfrak{M}$, $l: \mathfrak{M} \to \mathcal{P}$, with the same middle space $\mathfrak{M}$ and the same light factor l. The relations between the two equivalences have been discussed, especially in the case when P_1, P_2 are 2-manifolds, by J. W. T. Youngs [Ann. of Math. (2) **45**, 753–785 (1944); these Rev. **6**, 278]. F-equivalence always implies K-equivalence; K-equivalence does not imply in general F-equivalence, though there are a few cases where this implication can be proved, e.g. when P_1, P_2, $\mathfrak{M}$ are 2-cells, or are 2-spheres. In the present paper the author takes into account mappings T_1, T_2 from 3-dimensional manifolds P_1, P_2. The author proves by examples that even in the case when P_1, P_2 are 3-cells, K-equivalence does not imply F-equivalence and so it appears that in the 3-dimensional case the relations between K- and F-equivalence are weaker than in the 2-dimensional case. *L. Cesari* (Lafayette, Ind.).

Boothby, William M. The topology of the level curves of harmonic functions with critical points. Amer. J. Math. **73**, 512–538 (1951).

In this paper the author continues his study [same J. **73**, 405–438 (1951); these Rev. **13**, 149] of plane curve families regular except possibly at an infinite set of points each of which is a singularity of the type of a multiple saddle point. The principal results are that such a family is homeomorphic to the level curves of a harmonic function, that they are homeomorphic to the family of solutions of a system of differential equations $dy/dt=p(x, y)$, $dx/dt=q(x, y)$ and that the family admits a decomposition into a countable sum of subfamilies of curves each having the topological structure of the family of parallel lines $y=c>0$. These results generalize theorems due to W. Kaplan [Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 299–301; Trans. Amer. Math. Soc. **63**, 514–522 (1948); these Rev. **3**, 135; **9**, 606] in which the curve families are regular in the plane and answers a question posed in the second paper cited. *W. R. Utz.*

Myškis, A. D. I. On the concept of boundary. II. The definition of a boundary by means of continuous mappings. III. On the equivalence of certain methods of definition of a boundary. Amer. Math. Soc. Translation no. **51**, 52 pp. (1951).

Translated from (I) Mat. Sbornik N.S. **25(67)**, 387–414 (1949); (II) ibid. **26(68)**, 225–227 (1950); (III) ibid. **26(68)**, 228–236 (1950); these Rev. **11**, 382, 609.

Hanner, Olof. Solid spaces and absolute retracts. Ark. Mat. **1**, 375–382 (1951).

A space X is said to be solid if for any normal space Y, any mapping into X of any closed subset of Y can be extended to all of Y. A normal space X is an absolute retract with respect to normal spaces (ARN) if X is a retract of any normal space of which it is a closed subset. It is shown in this paper that a normal space is solid if and only if it is an ARN. Next, the concept of ARN is compared with that of AR: a separable metric space which is a retract of any separable metric space of which it is a closed subset. It is shown that a separable metric space is an ARN if and only if it is an AR and an absolute G_δ. The corresponding theorems for neighborhood retracts are also proved. An example is given of an AR which is not an ARN.

E. G. Begle (New Haven, Conn.).

Hanner, Olof. Some theorems on absolute neighborhood retracts. Ark. Mat. **1**, 389–408 (1951).

The general program of this paper is the examination of a number of topological properties which are necessary for a space (all spaces considered here are separable metric) to be an ANR (=absolute neighborhood retract), and the proofs that some of them are also sufficient. First it is shown that a necessary and sufficient condition for a space to be an ANR is that each point of the space have an open neighborhood which is an ANR. The homotopy extension theorem for a space X is that a homotopy between two mappings into X of a closed subset B of a space Y can be extended whenever one of the mappings can be extended. It is known that this theorem holds for any ANR, and in this paper it is shown that any locally contractible space for which it holds is an ANR.

A space Z is said to dominate a space X if there are two mappings $f: X \to Z$ and $g: Z \to X$ such that $gf: X \to X$ is homotopic to the identity mapping. Z α-dominates X, where α is a covering of X, if there is a homotopy, call it F, between gf and the identity such that for each $x \in X$ there is an element U of α containing $F_t(x)$, $0 \leqq t \leqq 1$. It is shown that any ANR, for any covering α, is α-dominated by a locally finite polyhedron. (By the first result mentioned above, a locally finite polyhedron is an ANR.) Conversely, it is shown that X is an ANR if for each of its coverings α, there is an ANR which α-dominates X. Some refinements of this theorem are also proved. Finally, a new proof, and a slight generalization, are given of a theorem due to J. H. C. Whitehead: If X_1 and X_2 are disjoint ANR's, φ a mapping into X_2 of a compact ANR $B \subset X_1$, and if X is the space obtained from $X_1 \cup X_2$ by identifying each point $x \in B$ with $\varphi(x)$, then X is an ANR. *E. G. Begle.*

Keesee, John W. On the homotopy axiom. Ann. of Math. (2) **54**, 247–249 (1951).

Eilenberg and Steenrod [Proc. Nat. Acad. Sci. U. S. A. **31**, 117–120 (1945); these Rev. **6**, 279] have listed seven axioms for cohomology theory, and proved a "uniqueness theorem": any cohomology theory which is defined on triangulable spaces and which satisfies these axioms must coincide with the usual cohomology theory for triangulable spaces. They have further shown that if an eighth axiom, the so-called "continuity axiom", is assumed, then any cohomology theory which is defined on compact Hausdorff spaces and which satisfies all eight axioms must coincide with the Čech cohomology theory. The continuity axiom states roughly that the cohomology groups of the limit space of an inverse mapping system are direct limits of the cohomology groups of the spaces of the mapping system. In this note, the author proves that one of the seven axioms, the "homotopy axiom," is not needed in the presence of the continuity axiom. To be precise, the homotopy axiom is a consequence of axioms 1, 2, and the continuity axiom.

W. S. Massey (Providence, R. I.).

Uehara, Hiroshi. On a homotopy classification problem. Kōdai Math. Sem. Rep. **1951**, 7–14 (1951).

The purpose of this paper is stated by the author in the introduction as follows: "It is reported that the homotopy classification problem of an n-dimensional finite connected

complex with a fixed decomposition into an arcwise connected topological space with $\pi_i(Y)=0$ for $n>i>1$ has been solved completely by several topologists, P. Olum [Bull. Amer. Math. Soc. 53, 922, 1132 (1947)], M. M. Postnikov [Doklady Akad. Nauk SSSR 66, 161–164 (1949)], and S. T. Hu [Acad. Sinica Science Record 2, 227–244 (1949)]. But details are not yet to hand in Japan. I also intend to impart a solution to this problem by the aid of Eilenberg-MacLane's cohomology group of abstract group and of Steenrod's cohomology theory with local coefficients." [The papers of Postnikov and Hu referred to are reviewed in these Rev. 11, 451; the reference to Olum is to some abstracts of results of which a complete exposition has not yet been published]. Although the final result obtained is similar to that obtained by Postnikov, it should be noted that this paper, unlike the note of Postnikov, gives proofs for all the theorems stated. S. T. Hu [loc. cit.] treated only the special case in which the group $\pi_1(Y)$ operates trivially on $\pi_n(Y)$.

W. S. Massey (Providence, R. I.).

Hilton, P. J. Calculations of the homotopy groups of A_n^2-polyhedra. II. Quart. J. Math., Oxford Ser. (2) **2**, 228–240 (1951).

The author computes the $(n+2)$-homotopy groups $(n\geqq 3)$ for an A_n^2 polytope; they are expressible in terms of H_n, H_{n+1}, H_{n+2}, Steenrod's squaring operation, and certain homomorphisms. The technique is similar to that of part I [same J. (2) 1, 299–309 (1950); these Rev. 12, 519].

J. Dugundji (Princeton, N. J.).

Burger, E. Über die Homotopietypen gewisser Polyeder. Math. Ann. **123**, 263–284 (1951).

Let K be a (connected) complex such that $\pi_i(K)=0$, $1<i<n$. Then it has been shown by Eilenberg and MacLane [Trans. Amer. Math. Soc. 65, 49–99 (1949); these Rev. 11, 379] that $H_n(K)$ is determined by $\pi_1(K)$, $\pi_n(K)$ (as a group with operators in $\pi_1(K)$), and a certain characteristic cohomology class $k^{n+1}\in H^{n+1}(\pi_1, \pi_n)$. Moreover, if K, K' are two such complexes with the same systems (π_1, π_n, k^{n+1}), and if $\bar{K}$, $\bar{K}'$ are their universal covers (so that there is a natural isomorphism $H_n(\bar{K})\approx H_n(\bar{K}')$), then there exist "cover-true" (deckbewegungstreu) chain-mappings $g: \bar{K}^{n+1}\to\bar{K}'^{n+1}$, $g': \bar{K}'^{n+1}\to\bar{K}^{n+1}$, inducing the given isomorphism of $H_n(\bar{K})$ with $H_n(\bar{K}')$ and such that gg' and $g'g$ are both cover-true chain-homotopic to the identity in dimensions $\leqq n$.

In the present paper the author proves that the homotopy type of a polyhedron K, all of whose homotopy groups vanish except π_1 and π_n, is determined by the system (π_1, π_n, k^{n+1}). Since, presumably, there are no finite-dimensional polyhedra of the given type, the result is effectively an algebraic characterisation of the $(n+1)$-type of a complex K such that $\pi_i(K)=0$, $1<i<n$. As such, it generalizes a result of J. H. C. Whitehead and MacLane [Proc. Nat. Acad. Sci. U. S. A. 36, 41–48 (1950); these Rev. 11, 450], who dealt with the case $n=2$. The proof consists of the realization of the chain-mapping g by means of a map f: $\bar{K}\to\bar{K}'$, which is cover-true and simplicial on a suitable subdivision of $\bar{K}$. The step-by-step process of building up the map f makes use of Hopf's "Homotopierand", which is essentially the (many-valued) mapping $\beta\omega^{-1}$: $H_r(L^r, L^{r-1})\to\pi_{r-1}(L^{r-1})$, where L is any complex, ω is the natural homomorphism of $\pi_r(L^r, L^{r-1})$ on $H_r(L^r, L^{r-1})$, and β is the boundary homomorphism $\pi_r(L^r, L^{r-1})\to\pi_{r-1}(L^{r-1})$. A similar argument is used to realize g' and the chain-homotopies $gg'\sim 1$, $g'g\sim 1$, although, as the author points out, the use of a theorem due to J. H. C. Whitehead leads almost immediately to the conclusion that the map $f_0: K\to K'$, induced by $f: \bar{K}\to\bar{K}'$, is a homotopy equivalence.

The paper closes by deriving an algebraic analogue of the problem of classifying maps of an arbitrary connected polyhedron into a connected polyhedron all of whose homotopy groups vanish except π_1 and π_n. The author acknowledges that the classification problem has been solved by Olum [Bull. Amer. Math. Soc. 53, 1132 (1947)] and Postnikov [Doklady Akad. Nauk SSSR (N.S.) 67, 427–430 (1949); these Rev. 11, 451]. *P. J. Hilton* (Manchester).

Burger, E. Über Schnittzahlen in Komplexen mit Automorphismen. Math. Ann. **122**, 131–143 (1950).

Let K be the barycentric subdivision of an n-dimensional simplicial complex and let $\mathfrak{G}$ be a group of simplicial mappings of K on itself such that no simplex is mapped on itself by more than a finite number of elements of $\mathfrak{G}$. It is assumed that K has a fundamental n-cycle that is invariant under $g\in\mathfrak{G}$, and that K has a finite fundamental domain with respect to $\mathfrak{G}$. Let * be a homomorphism of the integral group ring $\mathfrak{X}$ of $\mathfrak{G}$ into the ring $\mathfrak{X}^*$ of integers of an algebraic number field such as is induced by a homomorphism of $\mathfrak{G}$ into a group of roots of unity. The $\mathfrak{X}$-module $C^p(K)$ of p-cochains is thereby mapped into an $\mathfrak{X}^*$-module $C^{p*}(K)$. The resulting cohomology group $H^{*p}(K)$ is an $\mathfrak{X}^*$-module not necessarily a homomorph of $H^p(K)$. In a manner analogous to the definition of Reidemeister's τ-product [Monatsh. Math. Phys. 48, 226–239 (1939); these Rev. 1, 105] an intersection number for cocycles f_1 and f_2 of dual dimensions p and $n-p$ is defined by

$$S(f_1^*, f_2^*)=(\sum_{\sigma^n}\sum_{g}(f_2\cup gf_1)(\sigma^n)g)^*\in\mathfrak{X}^*.$$

The Betti modules $H_0^{*p}(K)$ and $H_0^{*n-p}(K)$ are a group pair with respect to the induced intersection relation. They are a dual pair in the important case: $\mathfrak{G}$ the group of covering transformations of a regular covering K of some complex K_0. Furthermore, in this case the intersection of cocycles is essentially the same as the intersection of the dual cycles previously defined by the author [Math. Z. 52, 217–255 (1949); these Rev. 12, 43]. The invariants of a dual group pair $\mathfrak{M}_1$ and $\mathfrak{M}_2$ determine the group pair; the ranks of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are equal, and the class of $\mathfrak{M}_1$ multiplied by the conjugate of the class of $\mathfrak{M}_2$ is equal to the class of the largest determinant divisor $\neq 0$ of the intersection matrix. This generalizes the author's previous generalization of Pontrjagin's duality theorem [loc. cit.]. If K is a decomposition of a locally-finite polyhedron $\mathfrak{P}$, and is such that whenever $g\in\mathfrak{G}$ maps a simplex of K on itself it maps also each face of the simplex on itself, then the groups $H^{*p}(K)$ and their intersection numbers are topological invariants of $\mathfrak{P}$. If, further, K is a regular covering of K_0 and $\mathfrak{G}$ is the group of covering transformations then the groups and the intersection numbers of K are topological invariants of K_0. *R. H. Fox.*

Burger, E. Zur Theorie der Überlagerungsabbildung von Mannigfaltigkeiten. Math. Ann. **122**, 144–151 (1950).

Let t be a simplicial mapping of an oriented closed n-dimensional manifold M into another oriented closed n-dimensional manifold μ. Let $\mathfrak{N}$ and Φ be the kernel and image respectively of the induced homomorphism of $\mathfrak{G}=\pi_1(M)$ into $\Gamma=\pi_1(\mu)$. The mapping t is covered by a mapping t_1 of M into μ_Φ, the covering of μ that belongs to Φ; t_1 is covered by a mapping t_2 of $M_\mathfrak{N}$, the (regular) covering of M that belongs to $\mathfrak{N}$, into the universal covering

μ_ϵ of μ. The mappings t_1 and t_2 have the same degree $c_1=c_2$; if the index j of Φ in Γ is finite $c_2 j=c$, the degree of t, and if j is infinite $c=c_2=0$. Let $\mathfrak{X}$ be the integral group ring of the group $\mathfrak{G}/\mathfrak{N}\approx\Phi$ (which is the group of covering transformations of the regular covering $M_\mathfrak{N}$ of M and also of the regular covering μ_ϵ of μ_0), let $\mathfrak{X}$ be mapped into the ring $\mathfrak{X}^*$ of integers of an algebraic number field as in the paper reviewed above. Let τ_2^* denote the t_2-induced homomorphism of $H^{*k}(\mu_\epsilon)$ into $H^{*k}(M_\mathfrak{N})$. The author proves a number of theorems of which the following are representative: if $c\neq 0$, $H_0^{*k}(\mu_\epsilon)$ is isomorphic to a subgroup of $H_0^{*k}(M_\mathfrak{N})$; if $c_2=\pm 1$ and rank $H_0^{*k}(M_\mathfrak{N})=$rank $H_0^{*k}(\mu_\epsilon)$ then $H_0^{*k}(\mu_\epsilon)$ is mapped isomorphically and product-true by τ_2^* onto $H_0^{*k}(M_\mathfrak{N})$; if $c_2=\pm 1$, the intersection matrix of μ_ϵ is divisible by that of $M_\mathfrak{N}$; if $c_2=\pm 1$, the torsion group $T^{*k}(\mu_\epsilon)$ is isomorphic to a subgroup of $T^{*k}(M_\mathfrak{N})$; etc. These theorems about t_2 are analogues of theorems proved by Hopf [J. Reine Angew. Math. 163, 71–88 (1930)] about t. *R. H. Fox.*

Smirnov, Yu. The Betti groups of the intersection of an infinite number of sets. Doklady Akad. Nauk SSSR (N.S.) 76, 29–32 (1951). (Russian)

The author sketches a proof (involving three subsidiary theorems) of the two following results. The intended application of these theorems is to a proof of the classical duality relations for general subsets of spherical n-space, or compact sets in euclidean n-space. Theorem 1: Let R be an arbitrary topological space and let ϕ_λ be a system of bicompacta ordered by inclusion. The Betti groups $\Delta^p\phi_\lambda$ taken with the homomorphisms of inclusion E_λ^μ generate an inverse spectrum whose limit group is the group $\Delta^p\phi$ of the intersection ϕ of the sets ϕ_λ. Theorem 2: Suppose that in a hereditarily normal space R there is given a set M. The Betti groups $\delta^p U_\lambda$ of neighborhoods U_λ of the set M taken with the homomorphisms of inclusion E_λ^μ generate an inverse spectrum whose limit group is the group $\delta^p M$. The groups Δ^p are based on finite coverings, and are taken with discrete or bicompact coefficients; the groups δ^p are groups of finite chains based on countable locally finite coverings, taken with discrete coefficients only. *L. Zippin* (Brooklyn, N. Y.).

Glezerman, M., and Pontryagin, L. Intersections in manifolds. Amer. Math. Soc. Translation no. 50, 149 pp. (1951).

Translated from Uspehi Matem. Nauk (N.S.) 2, no. 1(17), 58–155 (1947); these Rev. 10, 391.

de Rham, G. Complexes à automorphismes et homéomorphie différentiable. Ann. Inst. Fourier Grenoble 2 (1950), 51–67 (1951).

For a complex with a group G (cyclic of order h, generator γ) of automorphisms without fixed cells, the author gives a new definition of invariants he introduced [Comment. Math. Helv. 12, 191–211 (1940); these Rev. 2, 74], which generalize the Franz-Reidemeister torsion. Let C be the group of chains (complex coefficients) of the complex, with G operating on the complex. Then C is the direct sum of the groups C_ξ, with ξ running over the hth roots of unity, where C_ξ consists of all chains c satisfying $\gamma c=\xi c$. The boundary operator ∂ maps the group (actually vector space) C_ξ', i.e., the elements of odd dimension of C_ξ, into the vector space C_ξ'', the elements of even dimension, and conversely. This situation, a pair of vector spaces with mutual maps L, such that $L^2=0$ is considered in general; a certain function D is defined, involving the exterior products of sets of vectors in various sub- and quotient spaces. For the present situation, taking into account the fact that C has a preferred basis, namely, the cells themselves, one obtains a function $\Delta_\xi=\Delta(b_\xi'/b_\xi'')$, depending on elements of the homology groups B_ξ', B_ξ'', whose values are determined up to a factor $\pm\xi^q$. If B_ξ' and B_ξ'' are of rank 0, one can take b_ξ' and b_ξ'' equal to 1, and obtain a number (up to sign); this is the torsion. Invariance of the Δ_ξ under subdivision is proved. For a compact differentiable manifold V of class C^3 it is shown that the nerve of any geodesically convex (with respect to some ds^2) covering has the homotopy type of V. If a differentiable map T of period h, class C^3, without fix points, operates on V, one can consider invariant coverings, on whose nerves T also operates; it is shown that all these have the same invariants Δ_ξ. Moreover, if two such maps T, $\bar{T}$ are differentially equivalent ($T=f\cdot\bar{T}\cdot f^{-1}$), then they have the same invariants. During the proof it is shown that, if there exists such an f of class C^1, there exists also one of class C^2. The theory is applied to the sphere S^{2n-1}, with T a rotation. The invariants Δ_ξ determine the eigenvalues of T, and conversely. It follows that two rotations T, $\bar{T}$, which are differentiably equivalent, have the same eigenvalues, and so are isometric. As a similar application, if two Clifford-Klein spaces of the same positive curvature are differentiably equivalent, then they are isometric.

H. Samelson (Ann Arbor, Mich.).

GEOMETRY

Cavallaro, Vincenzo G. Notes sur la géométrie du triangle. Anais Fac. Ci. Porto 34, 5–22 (1949).

In one note the author considers the special triangle whose incircle passes through the centroid of that triangle. A number of quantitative relations are given for various remarkable elements of the triangle, particularly in terms of the in- and circum-radii of the triangle. The same is done, in a second note, for the case when the incircle passes through the center of gravity of the perimeter of the triangle. A third note is devoted to the triangle the sum of the squares of the sines of whose angles is equal to unity. This "harmonic" triangle is obtuse-angled, its area and the area of its orthic triangle are equal, etc. The author refers to an earlier paper [Giorn. Mat. Battaglini (3) 19(66), 24–30 (1928)] where he first considered this harmonic triangle. There are a number of disturbing misprints. *N. A. Court.*

Kárteszi, Ferenc. Über axialsymmetrische Moebius'sche Tetraederpaare. Mat. Lapok 2, 89–98 (1951). (Hungarian. Russian and German summaries)

If one considers a fixed tetrahedron T, then there are straight lines k such that the tetrahedron T', obtained by reflecting T on k, forms together with T a Moebius pair of tetrahedra. The paper deals with the problem of determining the locus θ of these axes of symmetry. Similarly, it is possible that a biaxial involution with axes p and q transforms a tetrahedron T into a tetrahedron T' such that T and T' form a Moebius pair. The author calls then p and q a pair of Moebius axes with respect to the given tetrahedron T. The following theorem is proved: A pair of skew lines p and q is a pair of Moebius axes for T if and only if there exists a hyperboloid H such that T is a polar tetrahedron of H while p and q are skew generators of H. By means of this

theorem it is possible to determine θ. It is shown that θ is the locus of the generators in the saddle points of all equilateral hyperbolic paraboloids which have T as a common polar tetrahedron. It is stated without proof that the ruled surface θ is of order nine. Finally, the case of tetrahedra with congruent faces is discussed in detail. *E. Lukacs.*

Bilo, J. Sur les homographies ayant des tétraèdres homologues formant un couple de Moebius. Mathesis **60**, 172–176 (1951).

Ilitsch-Dajovitsch, Militza. Ein Beweis des Descartes-Eulerschen Satzes. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 39–41 (1950). (Serbo-Croatian. German summary)

Zacharias, Max. Streifzüge im Reich der Konfigurationen: Eine Reyesche Konfiguration (15_3), Stern- und Kettenkonfigurationen. Math. Nachr. **5**, 329–345 (1951).

Following Reye [Die Geometrie der Lage, 5th ed., vol. 3, Kröner, Leipzig, 1923, p. 72], the author considers a conic and a plane cubic, intersecting in six points. Each pair of the six points is collinear with a third point on the cubic. The fifteen such points lie by threes on fifteen new lines, forming a regular configuration 15_3, different from the irregular 15_3 of the author's earlier paper [Math. Nachr. **3**, 243–256 (1950); these Rev. **12**, 523]. He is evidently unaware of the relevant investigations of Cremona, Martinetti, and Richmond [see Coxeter, Bull. Amer. Math. Soc. **56**, 413–455 (1950), p. 441; these Rev. **12**, 350]. He finds, among the fifteen points, six sets of five whose joins do not belong to the configuration. He also finds ten ways of dividing the points and lines into a double-three of lines and a double-three of points. Each of the nine points in the double-three of lines lies on a definite one of the nine lines in the double-three of points. He describes a remarkable metrical specialization having pentagonal symmetry. He rediscovers, as a degenerate case, Kantor's "non-Desargues" 10_3 [Coxeter, loc. cit., p. 438], which consists of two mutually inscribed pentagons. He generalizes the 15_3 and 10_3 to a $3n_3$ and $2n_3$ for any odd n greater than 1. When $n=3$, the former is the Pappus 9_3. The results when $n=7$ have a very agreeable appearance. Finally, he describes a 12_3 (different from the reviewer's) consisting of a cycle of four triangles, each inscribed in the next. *H. S. M. Coxeter.*

Niggli, Alfred, und Niggli, Paul. Raumgruppensymmetrie und Berechnungsmethoden der Kristallstrukturlehre. I. Z. Angew. Math. Physik **2**, 217–232 (1951).

Liegt der Betrachtung der Kristallstruktur eine primitive Zelle zugrunde, so werden darin durch die spezielle Symmetrie jedem Punkt gewisse identische Punktlagen zugeordnet. Deren Anzahl ist einzig von der betrachteten Symmetriegruppe abhängig. Um bei vorgelegter Gruppe die Verhältnisse zu übersehen, bedient man sich in der reinen Mathematik der Darstellungstheorie der Gruppen. In der vorliegenden Arbeit werden die Verhältnisse in einer den Bedürfnissen der Kristallographie angepassten Form vorgelegt. Dabei werden sowohl die Gruppen mit Fixpunkt wie auch diejenigen mit Translationen behandelt. Eine erste Tabelle gibt für alle 32 Klassen die Übersicht über die möglichen Achsen und übrige Symmetrieelemente, weitere Tabellen behandlen die Symmetrieverhältnisse in den einzelnen Klassen. *J. J. Burckhardt* (Zürich).

Eves, Howard, and Hoggatt, V. E., Jr. Hyperbolic trigonometry derived from the Poincaré model. Amer. Math. Monthly **58**, 469–474 (1951).

Moneo, Alberto. Double points of semihomographic transformations. Gaceta Mat. (1) **3**, 113–124 (1951). (Spanish)

Ničе, V. Construction d'une cubique gauche à l'aide des points imaginaires conjugués. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 35–38 (1950). (Serbo-Croatian. French summary)

Fernández Biarge, Julio. Invariants of plane sections of quadrics. Gaceta Mat. (1) **3**, 108–112 (1951). (Spanish)

Di Noi, Salvatore. Le congruenze sulla retta nella geometria proiettiva. Period. Mat. (4) **29**, 79–90, 127–141 (1951).

Petronijević, Branislav. Postulatensystem der n-dimensionalen Euklidischen Geometrie. Srpska Akad. Nauka. Zbornik Radova, Knj. **7**, Matematički Institut, Knj. **1**, 58–63 (1951). (Serbo-Croatian. German summary)

Versuch eines Postulatensystems der n-dimensionalen Euklidischen Geometrie. Zum Unterschied von früheren Versuchen dieser Art (Euklid, Hilbert, Thieme) werden vom Verfasser folgende drei Neuigkeiten eingeführt: 1. Abtrennung der Geometrie der eindimensionalen Geraden (Rectometrie) von derjenigen der zweidimensionalen Ebene (Planimetrie). 2. Erhebung des Begriffes der Strecke zu dem Grundbegriffe der Rectometrie. 3. Einführung der Verknüpfungspostulate in die n-dimensionale ($n \geqq 3$) Geometrie. *Author's summary.*

✓ **Bouligand, G. L'accès aux principes de la géométrie euclidienne. Introduction à l'axiomatique du plan.** Librairie Vuibert, Paris, 1951. viii+88 pp. 320 francs.

This is an excellent elementary introduction, on the level of the "classe de seconde", to modern geometry based on F. Klein's "Erlanger Programm". Starting with traditional theorems and construction and locus problems, the role of transformations is discussed, the notion of group is introduced, and the affine, metric, and similarity groups are studied. The logical aspects are stressed throughout the text, and Hilbert's axiomatic treatment of geometry is given in a final chapter. A collection of examples adds to the value of the text. *F. A. Behrend* (Melbourne).

Hohenberg, Fritz. Eine reelle Darstellung der Hyperkegelschnitte. Monatsh. Math. **55**, 146–152 (1951).

Ein komplexer Punkt P der Ebene liegt auf einer einzigen reellen Geraden g und lässt sich durch die elliptische Involution auf g darstellen, deren Festpunkte P und der konjugiert komplexe Punkt $\bar{P}$ sind; die (reellen) Potenzpunkte der Involution sind P_1 und P_2 und das geordnete Paar P_1P_2 kann als Bild des Punktes P aufgefasst werden. Verf. untersucht das Bild eines Hyperkegelschnittes K, das ist die Gesamtheit der komplexen Punkte (z_1, z_2), die einer Gleichung

$$u_{00}+(u_{01}+iv_{01})z_1+(u_{01}-iv_{01})\bar{z}_1+(u_{02}+iv_{02})z_2+(u_{02}-iv_{02})\bar{z}_2 +u_{11}z_1\bar{z}_1+(u_{12}+iv_{12})z_1\bar{z}_2+(u_{12}-iv_{12})\bar{z}_1z_2+u_{22}z_2\bar{z}_2=0$$

genügen, wo u_{kl} und v_{kl} reell sind. Die Arbeit gehört ein von Segre betrachtetem Gebiet an [vgl. Coolidge, The

geometry of the complex domain, Oxford, 1924; Hohenberg, Österreich. Akad. Wiss. Math. Nat. Kl. S.-B. IIa. 157, 177–236 (1949); diese Rev. 11, 679]. *O. Bottema.*

Perassi, Rinaldo. Sulla geometria elementare a tre dimensioni in un corpo finito. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 84, 189–207 (1950).

A plane π: $ax+by+cz+d=0$ in the Euclidean three space over a finite field K is classified according to the value of $Q=-(a^2+b^2+c^2)$. (1) If $Q=0$, π is isotropic. (2) If $Q\neq 0$ is not a square, π is of first type. (3) If $Q\neq 0$ is a square, π is of second type. A line perpendicular to π is given the same classification as π. Every segment of an isotropic line has zero length. Two lines can have congruent segments if and only if they are of the same type. There is exactly one isotropic plane through an isotropic line. A plane of first type contains no isotropic lines. A plane of second type contains two distinct isotropic lines through each point. In addition to further results of this character, the author also finds the number of points on a sphere or circle. *Marshall Hall.*

Ellis, David. Geometry in abstract distance spaces. Publ. Math. Debrecen 2, 1–25 (1951).

A survey of geometry in semimetric and metric spaces can be found in L. M. Blumenthal's "Distance geometries" [University of Missouri Studies, vol. 13, no. 2 (1938)]. The present paper expounds in condensed form, without proofs, the scattered results so far obtained in geometry of abstract distance spaces, i.e. of sets bearing a distance $d(x, y)$ borrowed from a set G, called "ground set", which may be a directed set, a groupoid, a group (Abelian or not), a field, a Boolean algebra. Abstract distance geometry originated in the Viennese Circle in 1930–1933 with Menger and Taussky. The theories of Weil's uniform spaces and of Fréchet's spaces "à écart abstrait" belong to it although they developped independently. The interest of the author lies more along the line of L. M. Blumenthal. He gives a lucid and extensive classification of notions, many of which, like "congruence", "superposability", "metric base", are immediate generalizations of the corresponding notions in the numerical case. Results are formulated regarding the following abstract distance spaces. (1) Naturally oriented groups (Menger, Kestelman and Smith, Taussky), i.e. additively written groups where $d(a, b)=b-a$. (2) Naturally metrized groups (Menger, Taussky, D. Ellis), i.e. groups where the norm of a is defined as the unordered pair $(a, -a)$. (3) Square-metrized vector spaces (Taussky), i.e. n-dimensional vector spaces over fields of characteristic zero where $d(p, q)=\sum(p_i-q_i)^2$. (4) Autometrized Boolean algebras (D. Ellis), i.e. Boolean algebras where $d(a, b)=$ symmetric difference of a and b. While many of the notions and problems are of geometric origin, the modern tendency of abstract distance geometry is towards algebra.

C. Y. Pauc (Cape Town).

Dirac, G. A. Collinearity properties of sets of points. Quart. J. Math., Oxford Ser. (2) 2, 221–227 (1951).

This paper is concerned with some extensions of a theorem conjectured by Sylvester and first proved by Grünwald [cf. Amer. Math. Monthly 51, 169–171 (1944), problem no. 4065] that a finite non-collinear subset of the real plane of points determines at least one line that is on exactly two of the points. The principal theorem of the paper establishes that there are always at least three such lines. There is proved, besides, the projective plane dual of the theorem, and an analogue for n points of three-space, no three collinear. [Note. The results of this paper are contained, as special cases, in a much more general investigation of the Sylvester theorem and its extensions given by Th. Motzkin [Trans. Amer. Math. Soc. 70, 451–464 (1951); these Rev. 12, 849].] *L. M. Blumenthal.*

Convex Domains, Extremal Problems, Integral Geometry

***Pólya, G., and Szegö, G. Isoperimetric Inequalities in Mathematical Physics.** Annals of Mathematics Studies, no. 27. Princeton University Press, Princeton, N. J., 1951. xvi+279 pp. $3.00.

L'ouvrage rassemble les resultats connues, dont beaucoup sont dûs aux auteurs, en reprenant, perfectionnant, développant des questions d'extrema géométriques, parfois anciennes et plus ou moins connues ou traitées auparavant: problème isopérimétrique des anciens, rigidité maxima à la torsion d'un cylindre de section d'aire fixée (atteinte pour la section circulaire selon des considérations intuitives de Saint-Venant), capacité électrostatique minima d'un corps de volume donné (réalisée pour la sphère selon des indications incomplètes de Poincaré), frequence fondamentale minima des membranes ou plaques encastrées d'aire donnée (Lord Rayleigh) et d'une forme plus ou moins imposée, etc. Ces questions reprises depuis quelques dizaines d'anneés par divers auteurs conduisent à beaucoup d'inégalités, dites ici isopérimétriques par extension du problème célèbre; elles sont d'ailleurs, vu la difficulté des calculs exacts, fort utiles pratiquement, par comparaison avec des cas particuliers pour lesquels on peut donner des formules explicites utilisables ou des résultats numériques et que les auteurs détaillent dans des tables.

On compare pour des domaines (toujours de frontière assez régulière pour se prêter au calcul différentiel et intégral élémentaire) des valeurs de diverses fonctionnelles telles que: dans l'espace, volume, aire de la frontière, capacité électrostatique, épaisseur moyenne; dans le plan, aire, longueur du périmètre, moment d'inertie, rayon intérieur r_a (rayon du cercle sur lequel on peut faire l'application conforme avec rapport de similitude unité en a d'image au centre) rayon extérieur analogue, capacitè dans l'espace, rigidité à la torsion, frèquence fondamentale. On introduit aussi les rayons des sphères ou cercles dont les fonctionnelles valent celles des corps considérés, ce qui permettra souvent une expression plus concise des comparaisons.

On étudie l'effet de certaines déformations, petities ou non, et en particulier de la symétrisation de Steiner (elle consiste dans l'espace, étant donné un plan, à remplacer l'ensemble des points du corps situés sur une perpendiculaire par un seul segment de cette perpendiculaire, de même longueur totale, et symétrique par rapport au plan).

La capacité d'un corps est beaucoup etudiée par deux méthodes donnant des valeurs majorantes et minorantes: le principe de Dirichlet consiste a voir que l'intégrale de Dirichlet de f prise hors du corps et s'annulant à l'infini, de valeur 1 sur le corps, est minima lorsque f est harmonique (potentiel d'équilibre); l'intégrale minima est à une facteur près la capacité. Le principe de Thomson consiste à voir que l'intégrale de $|\mathbf{V}|^2$ ($\mathbf{V}$ à divergence nulle, de flux donné autour du corps) est minima lorsque $\mathbf{V}$ est le gradient du potentiel d'équilibre; l'intégrale minima est à un facteur

près, l'énergie, c'est-à-dire, la charge étant donnée, l'inverse de la capacité. (Cela contient le "principe de Gauss".) On aura donc des nombres majorant C et $1/C$ en choisissant des f et V particuliers, c'est-à-dire des surfaces de niveau ou lignes de force. On étudie spécialement les corps convexes, étoilés, de révolution et les condensateurs plans.

Des principes analogues de minima d'intégrales fournissent des inégalités pour la rigidité à la torsion et la fréquence fondamentale d'un domaine plan. Des chapitres spéciaux concernent les corps presque circulaires ou sphériques, les plaques et membranes, les ellipsoïdes et lentilles, et des extensions de la symétrisation avec applications diverses.

M. Brelot (Grenoble).

Levi, F. W. On Helly's theorem and the axioms of convexity. J. Indian Math. Soc. (N.S.) Part A. **15**, 65–76 (1951).

Denote the intersection of all sets in a family F by ΠF. Suppose Γ is a family of sets such that (i): $\Pi F \in \Gamma$ whenever $F \subset \Gamma$. For each finite set $\{P_1, \cdots, P_n\}$ contained in some set of Γ, let $|P_1 \cdots P_n| = \Pi\{K \mid \{P_1, \cdots, P_n\} \subset K \in \Gamma\}$. Suppose Γ satisfies also (ii): If $m > n$ and $\{Q_1, \cdots, Q_m\} \subset |P_1 \cdots P_n|$, then there is a permutation $i_1, \cdots, i_m$ of $1, \cdots, m$, and a positive integer $r < m$ such that $|Q_{i_1} \cdots Q_{i_r}| \cap |Q_{i_{r+1}} \cdots Q_{i_m}| \neq 0$. The author proves the theorem H: If Γ satisfies (i) and (ii), $K_1, \cdots, K_N$ are sets of Γ all contained in $|P_1 \cdots P_n|$, and each n of the K_i's have a point in common, then $K_1 \cap \cdots \cap K_n \neq 0$. Since the convex sets in Euclidean spaces satisfy (i) and (ii), H includes Helly's theorem [Jber. Deutsch. Math. Verein **32**, 175–176 (1923)] on the intersection of convex sets. Now let S be any space which satisfies the axioms of incidence and of order (axioms I1–I6 and II1–II4 of Hilbert [Grundlagen der Geometrie, 7th ed., Teubner, Leipzig, 1930]) and call a set K in S "convex" if $B \in K$ is implied by A, $C \in K$ and B between A and C. It is proved that the family of all convex sets in S satisfies (i) and (ii), so H is applicable to S. Another application of H: If A is a free abelian group with n generators and G is a finite family of subgroups, each $n+1$ of which have a common element not the identity 1, then $\Pi G \not\subset \{1\}$. (On p. 73, line 9, read "$\not\subset$" for "$\supset$".) (For a related result, and other group-theoretical analogues of Helly's theorem, see R. Rado [J. London Math. Soc. **22**, 219–226 (1948), esp. p. 224; these Rev. **9**, 408].) In a final section the author adjoins to (i) and (ii) two other conditions on Γ and proves that if $|ABC|$ exists, $D \in |AC|$, and $E \in |BC|$, then $|AE|$ intersects $|AD|$.

V. L. Klee, Jr. (Princeton, N. J.).

Pogorelov, A. V. The rigidity of general convex surfaces. Doklady Akad. Nauk SSSR (N.S.) **79**, 739–742 (1951). (Russian)

Call convex surface the boundary of a convex body in E_3 with interior points. This paper announces the fundamental result: Two intrinsically isometric convex surfaces are congruent as sets in E_3. While details of the proof are missing the essential steps are outlined, and it seems that the complete proof will not be unwieldy.

Together with the author's result, that a realization of a regular line element is again regular [same Doklady (N.S.) **66**, 1051–1053; **67**, 791–794 (1949); these Rev. **11**, 201] and A. D. Alexandroff's result on the realizability as a convex surface in E_3 of an abstractly given two-dimensional convex metric, the present result settles a famous set of problems. The second of these problems is inherently connected with differential equations. Outside of Russia the first and the last problems have also been approached via differential equations with geometrically quite unacceptable restrictions. In addition to differentiability hypotheses which exclude such unnatural objects as cubes "with their sharp corners and overflat faces", the curvature is usually assumed to be positive. The present paper shows conclusively that this approach is inadequate. Unfortunately, papers of this type on rigidity and realizability continue to appear.

H. Busemann (Auckland).

Hadwiger, H. Der Würfel als Körper kleinster Relativoberfläche. Jber. Deutsch. Math. Verein. Abt. 1. **55**, 9–14 (1951).

Let W be the cube $|x_i| \leqq 1$ in E_n. The area F relative to W of the boundary of a bounded set A with exterior Jordan measure $|A|$ is defined by

$$F = \liminf_{\rho \to 0+} \{|A + \rho W| - |A|\} \rho^{-1}.$$

The Brunn-Minkowski theorem yields $F \geqq 2n|A|^{(n-1)/n}$. The present paper gives a direct elementary proof for this inequality.

H. Busemann (Auckland).

Fejes Tóth, László. Über gesättigte Kreissysteme. Math. Nachr. **5**, 253–258 (1951).

Suppose that n spherical caps, each of area f, are described on the surface of the unit sphere. New and simple proofs are given of the inequalities ($n \geqq 3$)

$$\frac{nf}{4\pi} \leqq \frac{1}{2}n\left\{1 - \frac{1}{2}\operatorname{cosec}\frac{n\pi}{6(n-2)}\right\} < \frac{\pi}{2\sqrt{3}},$$

$$\frac{nf}{4\pi} \geqq \frac{1}{2}n\left\{1 - \frac{1}{\sqrt{3}}\cot\frac{n\pi}{6(n-2)}\right\} > \frac{2\pi}{3\sqrt{3}}$$

previously obtained by the author [Comment. Math. Helv. **23**, 342–349 (1949); these Rev. **11**, 455]. The first inequality holds when no two spherical caps overlap, i.e. for the packing problem; the second holds in the covering problem when every point of the sphere's surface is covered by some cap. The proof makes use of Jensen's inequality.

A system of circles in the plane or on the unit sphere is called saturated if it is not possible to insert a circle of the same size in that part of the plane or sphere which remains uncovered, without overlapping the circles of the system. It follows easily that the density d of a saturated system satisfies $d \geqq \pi/6\sqrt{3} = 0.302\cdots$ for the plane, the inequality being the best possible. For n circles on the surface of the sphere the result is rather curious, since $d \geqq d_n$, where $d_1, d_2, d_4, \cdots$ is a decreasing sequence tending to $\pi/6\sqrt{3}$ and $d_2 = 1 - \frac{1}{2}\sqrt{2} = 0.292\cdots < \pi/6\sqrt{3}$. The following theorem is also proved. Suppose that a saturated system of n circles of spherical radius r is given on the unit sphere. Then the number N of similar circles which can be placed on the sphere without overlapping satisfies $N \leqq 3n$. Equality is attained only for $r = \frac{1}{4}$, $n = 2$, though the quotient N/n can approach 3 as closely as we please for sufficiently small values of r and $1/n$.

R. A. Rankin (Birmingham).

de Finetti, Bruno. Sulle stratificazioni convesse. Ann. Mat. Pura Appl. (4) **30**, 173–183 (1949).

Let the function $f(P)$ be concave for P in a convex affine space C; that is, let $f(P)$ satisfy $f(P) \geqq \sum \lambda_k f(P_k)$ for points P_k in C and for $P = \sum \lambda_k P_k$, $\sum \lambda_k = 1$, $\lambda_k > 0$. Then the inequality $f(P) \geqq c$ defines a one-parameter convex stratification S (continuous family of convex regions one inside the other) of C. The author investigates the converse question

of whether or not to a given convex stratification S of C there necessarily corresponds a concave function $f(P)$ related to S in the manner described above. He shows by examples of functions of two real variables that this is not the case; actually, there are counter-examples of functions of just one real variable. Under certain quite general conditions on S, however, such a concave function does exist. Thus if $\varphi(x, y)$ has bounded first and second order partial derivatives, then there is a concave function $f(x, y) \equiv F[\varphi(x, y)]$ whose level curves are the curves $\varphi(x, y) = c$. Methods are given for constructing solutions if they exist. It is shown that if there is a solution then there is a minimal solution assuming given values on two given level curves or surfaces; and if $f(P)$ is a minimal solution then the set of minimal solutions is given by $h + kf(P)$, where h and k are constants with $k > 0$. Applications to economics are indicated.

E. F. Beckenbach (Los Angeles, Calif.).

Algebraic Geometry

Muracchini, Luigi. Le varietà V_6 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. Boll. Un. Mat. Ital. (3) 6, 97–103 (1951).

The author lists the V_6 in S_r ($r \geqq 10$) whose tangent spaces form a variety of dimension $\leqq 9$. A detailed account of the work is to be published elsewhere. *J. A. Todd.*

Scott, D. B. On the fundamental theorem for point-point correspondences with valency on an algebraic surface. Pont. Acad. Sci. Acta 14 (1950), 57–61 (1951). (English. Latin summary)

If T is an algebraic correspondence of valency v on an algebraic curve, it is known that T^{-1} also has valency v. This result is also known for non-degenerate, irreducible, point-point correspondences in the sense of Albanese between algebraic surfaces only in the case $v = 0$. On a ruled surface F of order $n > 4$, the author constructs an irreducible, non-degenerate correspondence S which has valency -1 and is such that S^{-1} has valency $3 - n$. *H. T. Muhly.*

d'Orgeval, Bernard. La représentation des plans multiples et le nombre maximum de points doubles d'une surface algébrique. Ann. Inst. Fourier Grenoble 2 (1950), 165–171 (1951).

O. Chisini a donné en 1934 [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 688–693, 766–772 (1934)] une méthode pour obtenir des courbes de diramation d'un plan n-ple non dégénéré; les courbes en question s'obtiennent à l'aide d'une convenable variation continue à partir d'une courbe décomposée en $n-1$ courbes C_i d'ordres respectifs $1, 2, \cdots, n-1$, chacune comptée deux fois; la courbe C_i établit le passage du feuillet i au feuillet $i+1$ du plan n-ple; les points de diramation de la courbe C_i se trouvent dans les intersections de C_i avec C_{i-1} et C_{i+1} (s'il y a d'autres diramations sur C_i elles seront arbitraires). L'auteur se sert ici de cette construction en substituant la C_{n-1} double par une C_{2n-2} simple tangente à C_{n-2} partout où elle la rencontre; il obtient de cette façon un plan multiple limite, image d'une surface limite F^n d'ordre n; et il cherche d'imposer à cette surface des points doubles, soit en imposant des points doubles à la C_{2n-2}, soit en faisant coincider par couples les diramations des autres courbes C_i. Il arrive ainsi à trouver que les singularités de la surface limite F^n équivalent à un nombre maximum de points doubles compris entre $\frac{1}{8}(n^3+9n^2-40n+36)$ et la borne supérieure donnée par Basset en 1906. L'auteur même remarque que l'on ne peut pas affirmer que les singularités de la F^n limite puissent se résoudre en une quantité équivalente de points doubles isolés lorsque l'on passe par continuité de la forme limite à la forme générale. La même méthode peut s'appliquer à une surface d'ordre $2n$ possédant une conique n-ple.

E. G. Togliatti (Gênes).

Benedicty, Mario. Sopra le trasformazioni birazionali in sè di un campo neutro, in particolare nel caso di genere effettivo nullo. Ann. Scuola Norm. Super. Pisa (3) 4, 157–173 (1950).

The author considers a neutral field of characters p, δ_1, δ_2 (i.e. an algebraic curve C of effective genus p, on which δ_1 neutral pairs of distinct points and δ_2 neutral pairs of coincident points are fixed), whose virtual genus is therefore $\pi = p + \delta_1 + \delta_2$, and defines as birational transformations of the field into itself those birational transformations of C, which transform neutral pairs into neutral pairs; a field admitting such a transformation shall be called, for sake of brevity, field γ, and the aim of this paper is to study the fields γ. The main results (most of which are relative to the case $p = 0$) may be summarized as follows: (1) Schwarz-Klein theorem in a neutral field: A neutral field of virtual genus $\pi > 1$, different from the field χ, for which $p = \delta_1 = 0$, $\pi = \delta_2 = 2$, may admit only a finite number of birational transformations into itself. (2) Criterion for the construction of fields γ of effective genus $p = 0$: G being a finite group of projectivities on C (that is now a straight line), the possible pairs of γ are only the following ones, together with their transforms for the operations of G: (i) a generic pair of points; (ii) a generic pair of an involution of G; (iii) a pair of points, fixed for an operation of G; (iv) a generic pair of coincident points; (v) a pair of points, coincident in a fixed point of an operation of G. (3) Determination of the fields γ for $p = 0$: the field is obtained by the examination, for every finite group G, of all the possible systems of pairs transformed into themselves by the operations of G. (4) Theorems relative to the number ν of the transformations of γ; the most remarkable are: (i) for $p = 0$, $\pi > 1$, we have $\nu \leqq 12(\pi - 1)$, with the only exception of the above mentioned field χ, admitting infinitely many transformations; (ii) for $p = 0$, $\pi > 1$, we have $\nu \leqq 4\pi$, with 6 exceptions, five of which admit a finite number of transformations, and the sixth is given by χ. (5) Questions concerning the existence of fields γ, quasi-hyperelliptic or not, with $p = 0$ and arbitrarily fixed δ_1 and δ_2, and the consequent particularization of the moduli. (6) Analytic expressions of the transformations, by means of the normal integrals v, virtually of the first kind, of γ: the obtained expression is:

$$v'^{(\pi,1)} = \begin{Vmatrix} \alpha^{(\delta_1,\delta_1)} & 0 \\ 0 & \gamma^{(\delta_2,\delta_2)} \end{Vmatrix} v^{(\pi,1)} + \lambda_1^{(\pi,1)} \pmod{\omega},$$

where α is unimodular, ω is the period-matrix, and $\operatorname{Det} \gamma \neq 0$. (7) Transformations induced by the transformations of γ on Jacobi's quasi-abelian variety V: if u are the integrals of V, such a transformation is given by:

$$u' = \begin{Vmatrix} \alpha & 0 \\ 0 & \gamma \end{Vmatrix} u + \lambda \pmod{\omega}, \quad \text{with} \quad \lambda = \pi\lambda_1 \pmod{\omega}.$$

These transformations constitute a subgroup of the transformations considered by the reviewer [Ann. Mat. Pura Appl. (4) 27, 273–291 (1948); these Rev. 11, 391].

F. Conforto (Rome).

Samuel, P. Les méthodes d'algèbre abstraite en géométrie algébrique. Bull. Soc. Math. Belgique **3** (1949–1950), 11–17 (1951).

Expository lecture. Three directions for the study of the algebraic geometry are distinguished: absolute birational, birational restricted, and projective. The first is the study of the fields of algebraic functions without reference to any variety. The second treats the biregular birational correspondences. And the third is the study on a geometric model of the variety immersed in a projective space. [Reviewer's remark. The use of the term "point générique" for the concept of general point of van der Waerden seems inconvenient since it can give rise to confusion with the "punto generico" of the Italian School.] *P. Abellanas.*

Muhly, H. T. Integral bases and varieties multiply of the first species. Proc. Amer. Math. Soc. **2**, 576–580 (1951).

Let V be an absolutely irreducible r-dimensional variety defined by the H-ideal $\mathfrak{a}$ of the ring $R=k[X_0, \cdots, X_n]$, where k is a field of characteristic zero and X_i, $i=0, \cdots, n$, are indeterminates over k. Let $X_i=x_i(\mathfrak{a})$, $i=0, \cdots, n$, and $x_0, \cdots, x_r$, be an admissible set of independent variables for $\mathfrak{o}=k[x_0, \cdots, x_n]$ (i.e. $\mathfrak{o}$ is integrally dependent on $S=k[x_0, \cdots, x_r]$). Let $\mathfrak{a}_i=\mathfrak{a}+RX_1+\cdots+RX_i$ and $\chi(\mathfrak{a}, h)$ be the characteristic function of Hilbert [Nachr. Ges. Wiss. Göttingen **1889**, 25–34] associated with $\mathfrak{a}$ and $\Delta^i\chi(\mathfrak{a}, h)=\Delta^{i-1}\chi(\mathfrak{a}, h)-\Delta^{i-1}\chi(\mathfrak{a}, h-1)$, where $\Delta^0\chi=\chi$. The author proves: a necessary and sufficient condition for $x_0, \cdots, x_r$ to be an admissible set of independent variables for $\mathfrak{o}$ and for $\mathfrak{o}$ to possess an independent linear base over S [Muhly, Trans. Amer. Math. Soc. **54**, 340–360 (1943); these Rev. **5**, 74] is that $\Delta^i\chi(\mathfrak{a}, h)=\chi(\mathfrak{a}_j, h)$, $j=1, 2, \cdots, r+1$ for all values of h. These conditions are equivalent to the following: $\Delta\chi(\mathfrak{a}_i, h)=\chi(\mathfrak{a}_{i+1}, h)$, $i=0, 1, \cdots, r$, for all h, and these conditions state that $\mathfrak{a}$ is r times of the first species [Dubreil, Quelques propriétés des variétés algébriques . . . , Actualités Sci. Ind., no. 210, Hermann, Paris, 1935].

P. Abellanas (Madrid).

***Segre, Beniamino. Arithmetical Questions on Algebraic Varieties.** University of London, The Athlone Press, London, 1951. v+55 pp. 10/6d.

This booklet is based upon three lectures which the author gave at the University of London in 1950. It gives an account of problems and results in which algebraic geometry and number theory are interrelated. The topics treated represent chiefly the author's own contributions. For details of proofs the reader is referred to the original research papers, but the proofs are usually sketched and most of the necessary concepts and definitions are introduced in the text. The exposition is divided into three chapters. Chapter I deals with the theory of quadratic forms over an arbitrary commutative field. The topics include Dieudonné's results on the automorphisms of a quadratic form, representations between quadrics, and the determination of the linear spaces belonging to a quadric. In chapter II certain special types of Diophantine equations are studied by algebro-geometric methods (cubic and quartic surfaces, Severi-Brauer varieties, etc.). In chapter III the author studies some problems concerning rational varieties; gives applications to the Hilbert problem in the theory of resolvents; gives a partial solution of a problem (proposed orally by the reviewer) concerning the birational equivalence of hypercones; and discusses the arithmetical questions which come up when Severi's theory of the base is applied toward the determination of the birational transformation of an algebraic variety into itself. *O. Zariski.*

***Weil, André. Variétés abéliennes.** Algèbre et théorie des nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 125–127. Centre National de la Recherche Scientifique, Paris, 1950.

An abelian variety is a complete abstract variety which is a group variety; for instance, the Jacobian variety of a curve is an abelian variety. Let V be an algebraic variety without singularity in a projective space over the field of complex numbers; making use of the theory of simple integrals of the first kind on V, it is possible to associate to V an abelian variety A and a rational mapping f of V into A with the following properties: $f(V)$ is not contained in any proper abelian subvariety of A or in any translation of such a variety; any rational mapping g of V into an abelian variety B is composed of f and of a rational mapping h of A into B (h is then necessarily composed of a translation in A and a group homomorphism of A into B). If now V is any algebraic variety, then the author says that an abelian variety A is a variety of Albanese for V if there exists a rational mapping f of V into A with the properties stated above. The author states that he has the sketch of a proof for the existence of an Albanese variety for any variety V, but that his argument is not yet complete. The method to establish the existence of A depends on the construction of the Picard variety of V: this is an abelian variety whose points are in one-to-one correspondence with the classes of divisors on V which are algebraically equivalent to 0. The existence of the Picard variety P being assumed, the variety of Albanese A would be the Picard variety of P. The varieties A and P would then be in a kind of dual relationship, each one admitting a homomorphism onto the other; and this new kind of duality would have the logical features of the usual duality between groups (correspondence between subgroups on the one side and factor groups on the other, etc.; here the subgroups to be considered are naturally the algebraic ones).

The consideration of rational mappings of a variety V into an abelian variety A is, according to the author, an adequate substitute in the abstract case for the use of the integrals of the first kind (of simple differentials) for varieties over the field of complex numbers. Similarly, the algebraic substitute for integrals of differentials of the second or the third kind would be the consideration of rational mappings of a variety V into varieties of abelian groups which are not complete, but are fibered varieties [cf. the paper "Fibre-spaces in algebraic geometry", by A. Weil, Abstracts of the conference on algebraic geometry and algebraic number theory, University of Chicago, 1949] admitting an abelian variety A as base space, the fiber being a group which is either the multiplicative group G_m or the additive group G_a of the basic field.

It is stated that the fibered varieties whose fiber is G_m and whose base space is a variety U can be characterized by an invariant X, which is a class of equivalence of divisors on U. If U is an abelian variety, then the fibered variety corresponding to a given invariant X has a group structure if and only if the divisors in X are algebraically equivalent to 0.

C. Chevalley (New York, N. Y.).

✓*Zariski, O. **Quelques questions concernant la théorie des fonctions holomorphes sur une variété algébrique.** Algèbre et théorie des nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 129–133. Centre National de la Recherche Scientifique, Paris, 1950.

Let V/k be an irreducible algebraic variety defined over a field k and let W/k be a subvariety of V, also defined over k. The author introduces the notion of holomorphic function on V along W. This is a function which assigns to every point Q of W an element of the completion of the local ring of V at Q in such a way that the following condition be satisfied: W can be covered by a finite number of open sets (i.e. of complementary sets of subvarieties) G_i in such a way that, on each G_i, f may be approximated uniformly by a sequence of rational functions on V. The holomorphic functions form a ring $\mathfrak{o}^*(W)$. The main theorem of the theory states that this ring is a birational invariant. More precisely, assume that V/k is the image of another variety V'/k under a rational mapping T on which the following assumptions are made: if P' is a generic point of V', then $k(T(P'))$ is algebraically closed in $k(P')$; if $W'=T^{-1}\{W\}$ (the total transform of W by T^{-1}), then T is semi-regular at every point Q' of V' (i.e. if $Q\in W$ is any point which corresponds to Q', then the local ring of Q in V is contained in the local ring of Q' on V'). Assume finally that V is locally normal at every point of W. Then the rings $\mathfrak{o}^*(W)$ and $\mathfrak{o}^*(W')$ are isomorphic to each other. The proof of this theorem has been published in another paper by the author [Mem. Amer. Math. Soc., no. 5 (1951); these Rev. 12, 853].

As a first application of the main theorem, the author proves the generalization to the abstract case of the degenerescence principle of classical algebraic geometry. The link between this principle and the theory of holomorphic functions is established by the following connectedness criterion: if V is analytically irreducible at every point of W, then W is connected if and only if $\mathfrak{o}^*(W)$ is an integral domain. Making use of this and of the invariance of $\mathfrak{o}^*(W)$, the author establishes the following result. Let T be an irreducible algebraic correspondence between two varieties V and V'; we assume that V, V', T are defined over some field k and that, (P, P') being a generic point of T, $k(P)$ is quasi-algebraically closed in $k(P, P')$. Let Q be a point of V at which V is analytically irreducible; then the total transform $T\{Q\}$ of Q is connected over the field $k(Q)$. The principle of degenerescence is a special case of this.

Other applications are made to the theory of exceptional varieties. A subvariety W of V is called exceptional if there exists a rational mapping T of V onto a variety V'/k with the following properties: $W_0'=T\{W\}$ is zero-dimensional, T is semi-regular at every point of W and $T^{-1}\{W_0'\}=W$. In what follows, V is assumed to be normal. The author first proves that the connected components of any exceptional variety are themselves exceptional. If W is any subvariety of V, denote by $\mathfrak{o}(W)$ the ring of rational functions on V which are holomorphic along W (such functions must be constant on every connected component of W), and by $\mathfrak{m}$ the set of functions in $\mathfrak{o}(W)$ which are null on W. Let $H(W)$ be the set of points Q of V such that every function in $\mathfrak{m}$ is holomorphic at Q and takes there the value 0. Assuming W to be connected, the author proves that W is exceptional if and only if the following two conditions are satisfied: $\mathfrak{o}(W)$ contains non-constant functions, and $H(W)=W$.

Let now W be any subvariety of V. Define $\mathfrak{o}(W)$ as above. Then the author makes the following two conjectures: a) $\mathfrak{o}(W)$ is Noetherian, and therefore semi-local; b) $\mathfrak{o}^*(W)$ is the completion of $\mathfrak{o}(W)$. These conjectures are certainly true in the case where W is exceptional. Denote by r the dimension of V and by s the transcendance degree of the field of quotients of $\mathfrak{o}(W)$ (W being assumed to be connected). Then it is proved that, if the conjecture a) is true for varieties of dimensions $<r$, then it is true for varieties of dimension r and subvarieties W such that $s<r$. Assume now that $s=r$. Then it is proved that a) is true if the set $H(W)$ defined above is a subvariety of V. The assertion that $H(W)$ is a variety is proved to follow from the following conjecture on linear systems: if an irreducible subvariety Z of V is fundamental for all multiples of a linear system C, then, for n large enough, nC has no base point on Z (Z is called fundamental for C if, C_1 and C_2 being any two divisors in C not both containing Z, and x a rational function such that $(x)=C_2-C_1$, x is necessarily constant on Z). For varieties V without any singularity, the author conjectures that the following stronger statement is true: if C is a complete linear system, of dimension >0, without any fixed component, then, for n large enough, the complete linear system generated by nC has no base point. This conjecture is true in the case where V is a surface. *C. Chevalley.*

Koizumi, Shoji. On the differential forms of the first kind on algebraic varieties. II. J. Math. Soc. Japan 2, 267–269 (1951).

Using the terminology and methods (specialization rings) of A. Weil the author proves among other results: I. A differential form of the first kind on a product Variety $U\times V$ has the form $\sum\sigma_i\tau_i$ where σ_i, τ_i are differential forms of the first kind and complementary degrees on the components U, V, respectively. II. A differential form of the first kind on a complete Variety without singularities induces on each simple subvariety a differential form of the first kind.

O. F. G. Schilling (Chicago, Ill.).

Differential Geometry

✓*Blaschke, Wilhelm. **Einführung in die Differentialgeometrie.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LVIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1950. vii +146 pp.

This book presents the theory of curves and surfaces in Euclidean three-space. A previous book by the same author [Vorlesungen über Differentialgeometrie, vol. 1, 4th ed., Springer, Berlin, 1945] treats the same subject. The book reviewed here differs from the "Vorlesungen" mainly in that Cartan's method of the "repère mobile" together with Pfaffians have been used for both curves and surfaces. As a result the theory of surfaces and, in particular, the geometry on a surface (chapter IV) have been given a more elegant form than in the "Vorlesungen".

The size of the present book (146 pages) is considerably smaller than that of the "Vorlesungen" (304 pages). This is partly due to the fact that Cartan's method permits a shorter treatment of surfaces than older methods, partly to somewhat condensed discussions of various topics (however without material harm to clarity or readibility), and partly

to the fact that several topics taken up in the "Vorlesungen" have been either omitted (for instance, the chapters on "Invariante Ableitungen" and "Line geometry", occurring in the "Vorlesungen", have been dropped) or else rearranged (for instance curves are treated as "Schmiegstreifen" after the theory of strips has been developed). As in the Vorlesungen several problems concerning the behavior of curves and surfaces in the large are discussed.

The following is a list of chapters. I: Vectors, determinants, matrices (all in three-space). II: Strips ("Streifen") and curves (isotropic curves are excluded here but are discussed in the chapter on minimal surfaces). III: Pfaffians (mainly those in two variables). IV: "Interior" theory of surfaces (i.e. geometry on a surface). V: Geodesics. VI: "Exterior" theory of surfaces (i.e. the properties of a surface which are related to the Euclidean three-space in which it is imbedded). VII: Minimal surfaces. The author's reputation as a geometer and as a writer of textbooks is reaffirmed by this excellent work. *W. van der Kulk.*

Zavertanik, Aurelio. Relazione fra alcune classi di curve piane notevoli. Ann. Triestini. Sez. 2 (4) 3(19) (1949), 7–28 (1950).

Soit P un point de la courbe plane C, P_1 le centre de courbure, P_2 le centre de courbure de l'évolute, $P_1P_2=r$, P_3 le point sur la droite P_1P_2 de sorte que $P_1P_3=nr$. L'auteur considère les courbes $C(n)$ pour lesquelles PP_3 passe par un point fixe donné U. Quand U est à l'infini $C(n)$ est une courbe de Ribaucour. Équations intrinsèques de $C(n)$; équations paramétriques de $C(n)$. Cas spéciaux. La classe des courbes considérés contient la droite, le cercle, la cycloïde, la parabole, la caténoïde, la lemniscate, etc. *O. Bottema.*

Wrtilek, Franz. Tangenten- und Krümmungskreiskonstruktionen an ebenen Kurven mittels Deutung eines Kurvenparameters. Monatsh. Math. 55, 215–228 (1951).

A curve in the plane is described by the parametric equations $x=x(u)$ and $y=y(u)$. If altitudes of height u are erected normal to the plane of the curve, the ends of these altitudes describe a space curve. If the parameter u is taken as time, the slope of the space curve at a point is the instantaneous velocity through the corresponding plane point. If tangent vectors equal to the space velocities are drawn, their ends describe the first derivative curve whose projection is the hodograph of the given plane curve. Higher derivative curves are obtained in an analogous manner. As examples of the use of this representation, tangents and radii of curvature of a parabola are constructed; the motions, velocities and accelerations of a crosshead mechanism are studied. *M. Goldberg* (Washington, D. C.).

Jackson, S. B. A class of spirals. Duke Math. J. 18, 673–682 (1951).

Let $0<a\leqq+\infty$. Suppose $f(t)$ is continuous and $g(t)$ is continuously differentiable for $0\leqq t<a$; $f(t)>0$ if $t>0$ and $\lim_{t\to a} g(t)=\infty$. Finally suppose that $g'(t)/f(t)$ is strictly monotonic and $\lim_{t\to a} g'/f=0$ or ∞. The author discusses the geometric properties of the curve $z(t)=\int_0^t f(u)e^{i\omega(u)}du$ in the complex plane. It is a spiral without double points or vertices. If $t\to a$, then $z(t)$ either converges to a finite or infinite point or its limit points fill a straight line. In the latter case, the curve and its osculating circles lie entirely in one of the half-planes bounded by that straight line. *P. Scherk* (Saskatoon, Sask.).

Krzyż, Jan. On a certain converse of the mean value theorem. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 131–135 (1950). (French. Polish summary)

When the law of the mean in differential calculus is suitably stated, it admits the following converse: If a Jordan curve in three-dimensional Euclidean space has a tangent vector at each point and is such that each arc A of the curve contains a point P at which the tangent is parallel to the chord joining the end-points of A, then the curve lies in a plane.

The above result is easy to prove under the usual classroom assumptions of differential geometry. More generally, the author drops the assumption that P is a point of A and proves that if a curve in three-dimensional space is defined by continuous and differentiable functions, has a continuous tangent vector, and is such that the cone of chord directions is contained in the cone of tangent directions and the spherical indicatrix of tangents is of zero two-dimensional measure, then the curve lies in a plane. *E. F. Beckenbach.*

Pylarinos, O. Sur une relation entre les courbes sphériques et les courbes de Bertrand. Bull. Sci. Math. (2) 75, 91–96 (1951).

An investigation is made of pairs of curves with the property that the tangents to the first are parallel to the corresponding principal normals to the second. If the first curve is a Bertrand curve, the second is a spherical curve. To any spherical curve there corresponds in this fashion an infinity of Bertrand curves. *C. B. Allendoerfer.*

Fabricius-Bjerre, Fr. Über zykloidale Kurven in der Ebene und im Raum. Danske Vid. Selsk. Mat.-Fys. Medd. 26, no. 9, 75 pp. (1951).

Making use of some unpublished manuscripts by Hjelmslev, the author studies plane and skew cycloidal curves k. His discussion is based on the following characteristic property: The tangent at a variable point P of k intersects a fixed circle (sphere) in two real or imaginary points P_1, P_2 such that PP_1: PP_2 remains constant. Let O denote the center of the circle (sphere); α=square of its radius; M=midpoint of P_1P_2, $\mu=MP^2/MP_1^2$ (=const.; $\mu\neq 0$ or 1). Then this definition is equivalent to $MP^2+\mu\cdot OM^2=\mu\alpha$.

In the plane case, the k's are either hypocycloids ($\mu>1$, $\alpha>0$), epicycloids ($0<\mu<1$, $\alpha>0$), hypercycloids ($\mu<0$, $\alpha>0$), paracycloids ($\mu<0$, $\alpha<0$) or logarithmic spirals ($\mu<0$, $\alpha=0$). The evolute and the evolvents of k and the one-parametric family of similarities connected with k and its evolute are discussed. "Dann wird die Relation zwischen dem Radiusvektor und der Bogenlänge gewonnen, und in den folgenden Paragraphen werden die Polaren der zykloidalen Kurven, die sogenannten Ährenkurven sowie Kurven, die in anderer Weise mit den zykloidalen Kurven verknüpft sind, untersucht. Schliesslich werden diese verschiedenen Kurven zu einer passend gewählten nichteuklidischen Metrik in Verbindung gesetzt".

The definition of the skew k's leads at once to the following criterium: A skew curve is cycloidal if and only if it is mapped onto a plane k if a suitable cone through it is developped into the plane. This remark yields a classification of the skew k's. If more than one such cone exists, then k is called polycycloidal. The vertices of all these cones then fill a straight line. "Zu dieser Kurvenklasse [of the polycycloidal k's] gehören u.a. die polykonischen Loxodromen und diejenigen Böschungslinien auf einer Rotationsfläche 2. Grades, deren Tangenten einen festen Winkel mit der Achse bilden. Es wird eine Übersicht über die verschiedenen polyzykloi-

dalen Kurven gegeben und jeder einzelne Typus wird näher untersucht." *P. Scherk* (Saskatoon, Sask.).

Jonas, Hans. Klassen von viergliedrigen Zyklen Laplacescher Transformationen und Umhüllungsgebilde einer bewegten Dupinschen Zyklide. Math. Nachr. **5**, 259–300 (1951).

This paper deals with classes of cycles of four Laplace transformations and the envelopes of a moving Dupin cyclide. The seven sections contain: (1) a general involutary transformation for this kind of cycle; (2) nets of lines of curvature, nets of evolutes and the Laplace transform; (3) a preliminary solution of two problems concerning cycles containing a net of lines of curvature or a normal congruence; (4) the two cycles belonging to the bending surface of a hyperbolic paraboloid; (5) the H-surface determined by the bending of an orthogonal hyperboloid and its Laplace cycles; (6) generalized H-surfaces; (7) solutions of a problem concerning enveloping surfaces. *J. A. Schouten* (Epe).

Bompiani, Enrico. Über die Veronesesche Fläche. Arch. Math. **2**, 354–359 (1950).

A theorem is proved which is a sort of differential-geometric analogue of the theorem of Del Pezzo regarding the type of surface whose tangent planes are mutually incident. It reads as follows: If a surface is such that the tangent plane at each point P is cut by all planes tangent at points in a second order neighborhood of P, then it is either (a) a surface in S_4, (b) a developable, or (c) the surface of Veronese in S_5. *J. L. Vanderslice* (College Park, Md.).

Bompiani, Enrico. Sopra una nozione di spazio osculatore ad una varietà introdotta da L. Berzolari. Boll. Un. Mat. Ital. (3) **5**, 213–218 (1950).

The late L. Berzolari defined a certain metrically invariant "osculating" S_{m+1} containing the tangent S_m at a point of a V_m in a euclidean or constant curvature S_n [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **33**, 692–700, 759–778 (1898)]. The present writer generalizes this idea to the case where S_n is projective or even topological.

J. L. Vanderslice (College Park, Md.).

Bell, P. O. Hypergeodesic polygons whose ratio of angular excess to integral curvature is constant. Univ. Nac. Tucumán. Revista A. **7**, 141–156 (1950).

The above title defines the meaning of the term hypergeodesic as used herein. The main body of the paper develops a wide variety of theorems involving this family of hypergeodesics G_k. For example, the locus of the endpoints of the geodesic curvature vectors of the members of G_k at any given point is a circle. Also, the geometric nature of the envelope of osculating planes of G_k at a point is studied. Independently of G_k a new and simple characterization of the Gaussian curvature of a surface is obtained which lends itself to n-dimensional generalization. *J. L. Vanderslice.*

Hohenberg, Fritz. Eine Verallgemeinerung der Lilienthalschen Flächenpaare. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1951**, 129–131 (1951).

Pinl, M. Über einen Satz von L. Berwald und die Gausssche Krümmung der Minimalflächen. Monatsh. Math. **55**, 188–199 (1951).

Consider a two-dimensional complex surface immersed in n-dimensional Euclidean space R_n. If the isotropic curves are introduced as parametric curves, many formulas of surface theory are greatly simplified. On a boundedly isotropic curve it has been shown by Study and Vessiot that an invariant parameter may be introduced. It is advantageous to introduce these parameters as surface parameters when possible. The author proceeds to establish a theorem of L. Berwald that a surface in R_3 admits the Study-Vessiot parameters only if the surface has constant mean curvature. This condition is not sufficient even in R_3, and in R_n for $n>3$ it is not even necessary, as is shown by an example in R_{12} of a surface admitting the Study-Vessiot parameters but not having constant mean curvature. For a minimal surface in R_n the mean curvature is always zero, so the above condition is automatically satisfied. Moreover, a minimal surface is always a translation surface of isotropic curves. The paper then discusses minimal surfaces, primarily in R_4, from the point of view of the introduction of these parameters. It is established, for example, that among the complex minimal surfaces of R_4 having plane nonlinear translation isotropic curves, there are none with constant non-vanishing Gaussian curvature. *S. B. Jackson* (College Park, Md.).

Hsiung, Chuan-Chih. A general theory of conjugate nets in projective hyperspace. Trans. Amer. Math. Soc. **70**, 312–322 (1951).

The author establishes, with quite long computations, the three following theorems. I) In a linear space S_n of n ($\geqq 3$) dimensions let N_n be a conjugate net and π be a fixed hyperplane; then the points M, $\bar{M}$ of intersection of the fixed hyperplane π and the two tangents at a point x of the net N_n describe two conjugate nets N_M, $N_{\bar{M}}$ in the hyperplane π, respectively, and one of the two nets N_M, $N_{\bar{M}}$ is a Laplace transformed net of the other. II) In a linear space S_n of n ($\geqq 4$) dimensions let N_n be a conjugate net and S_{n-1} be a fixed linear space of $n-2$ dimensions; then the point T of intersection of the fixed subspace S_{n-2} and the tangent plane at a point x of the surface sustaining the net N_n describes a conjugate net N_T in the subspace S_{n-1}. III) Conjugate nets with equal and non-zero Laplace-Darboux invariants in a linear space S_n of n ($\geqq 4$) dimensions are characterized by the fact that at each point x of any one of them there exists a proper hyperquadric (and therefore $\infty^{\frac{1}{2}n(n+3)-11}$ such hyperquadrics) having second order contact at the Laplace transformed points x_{-1}, x_1 of the point x with both Laplace transformed surfaces S_{-1}, S_1 of the net N_n, respectively. It must be remarked, however, that theorems I and II are actually particular cases of very old results found by Bompiani [(a) Rend. Circ. Mat. Palermo **34**, 383–407 (1912); (b) C. R. Acad. Sci. Paris **156**, 603–605 (1913); (c) Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fiz. Mat. Nat. (5) **24**, 190–197 (1915)]. In fact, on page 604 of (b) it is proved that the ∞^2 spaces S_{h+k} joining the osculating S_h and S_k to the curves of a conjugate net of S_n at a same point may be considered (in two different ways) as osculating S_{h+k} to ∞^1 curves; therefore, their intersections with a S_{n+h-k} form also a conjugate net. If we take now $h=0$, $k=1$ (or $h=1$, $k=0$) we have, substantially, the author's theorem I (except for a corollary that is easily derived by geometric considerations); while for $h=k=1$, we have exactly theorem II of the present paper. As for theorem III, it is not very clear (at least to the reviewer) what the author means by "proper hyperquadric"; if he means "non-degenerate", the theorem is not correct for $n>4$; in fact, it is very easy to prove that the hyperquadrics osculating S_{-1} and S_1 at x_{-1}, x_1 are subject to $n(n+3)-12$ conditions; and all these hyperquadrics contain the plane (x, x_1, x_{-1}) at x; therefore, only

for $n=4$, these hyperquadrics are necessarily cones; for any n, however the above named conditions reduce to $n(n+3)-11$ if and only if N_x has equal (non-zero) Laplace-Darboux invariants; and the really important fact is this reduction of the number of the conditions imposed on the involved hyperquadrics. *V. Dalla Volta* (Rome).

Bartsch, Helmut. Übertragung der Achtflachgewebeeigenschaften auf Hyperflächengewebe des n-dimensionalen Raumes. Abh. Math. Sem. Univ. Hamburg **17**, 1–21 (1951).

Let $n>2$; let R_k denote a linear k-space $[k\leqq n]$; let G, G^* be regions in R_n, and let M_{n-1} denote $(n-1)$-dimensional hypersurfaces in G. Given a cartesian coordinate system in R_n, suppose G^* is convex with respect to the R_k's parallel to the coordinate-k-spaces $[k=1, 2, \cdots, n-1]$ and G^* is mapped on G topologically. Then the n families of M_{n-1}'s on which the intersections of G^* with the R_{n-1}'s parallel to the coordinate-$(n-1)$-spaces are mapped are said to form an M_{n-1}-net. An M_{n-1}-web consists of $n+1$ families of M_{n-1}'s such that any n of them form an M_{n-1}-net. Let $i, i', \cdots$ denote M_{n-1}'s of the ith family $[i=1, 2, \cdots, n+1]$. Suppose the $n+1$ surfaces $1, 2, \cdots, n+1$ determine the n-dimensional simplex S_n. Then there are M_{n-1}'s $1', 2', \cdots, n'$ such that their intersections with $n+1$ determine an $(n-1)$-dimensional simplex inscribed into the $(n-1)$-dimensional simplex $S_n \cap (n+1)$. The M_{n-1}-web is called a $(2n+2)$-cell-web if to every such construction an $(n+1)'$ exists such that

$$1' \cap i,\ 2' \cap i,\ \cdots,\ (i-1)' \cap i,\ (i+1)' \cap i,\ \cdots,\ (n+1)' \cap i$$

determine a simplex inscribed into $S_n \cap i$ $[i=1, 2, \cdots, n]$. A regular $(2n+2)$-cell-web consists of the $n+1$ families of $R_{n-1} \cap G$'s parallel to the sides of some regular simplex in R_n.

The author proves: An M_{n-1}-web is a $(2n+2)$-cell-web if and only if it is the topological image of a regular $(2n+2)$-cell-web. Thus any fixed M_{n-1} of a $(2n+2)$-cell-web is intersected by the M_{n-1}'s of the other families in an n-cell-web. If each M_{n-1} of a $(2n+2)$-cell-web is an $R_{n-1} \cap G$, then there exists a $(\binom{n}{2}-2)$-parametric family of quadrics in R_n whose common tangent-R_{n-1}'s coincide in G with the M_{n-1}'s of the web. The case $n=3$ is discussed in §5, pp. 42–50 of the following book: Blaschke and Bol, Geometrie der Gewebe . . . , Springer, Berlin, 1938. The author's results generalize the theorems of that section. *P. Scherk.*

Rollero, Aldo. A proposito di alcuni riferimenti intrinseci per le trasformazioni puntuali fra piani proiettivi. Atti Relaz. Accad. Pugliese Sci. Parte I. N.S. **7**, 195–204 (1949).

The author develops the proofs of some results already stated in a previous paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fiz. Mat. Nat. (8) **6**, 213–216 (1949); these Rev. **11**, 210], in which he determined an intrinsic coordinate system and a canonical form for a one-to-one transformation between two projective planes; moreover, in the present paper, a method of finding other intrinsic coordinate systems is briefly sketched. *V. Dalla Volta* (Rome).

Bell, P. O. A new approach to the study of contact in the projective differential geometry of surfaces. Duke Math. J. **18**, 689–695 (1951).

L'auteur applique à l'étude du contact des courbes et des surfaces une méthode tensorielle introduite par lui dans un article précédent [Trans. Amer. Math. Soc. **60**, 22–50 (1946); ces Rev. **7**, 529]. Un tétraèdre mobile de référence est attaché au point générique de la surface analytique S sur laquelle est tracé un réseau (u, v); une surface algébrique Σ associée à ce point générique est définie par son équation en coordonnées locales $f(x^0, x^1, x^2, x^3, u, v)=0$. On introduit des dérivées intrinsèques par rapport à u et v et des dérivées intrinsèques le long d'une courbe C de S qui permettent d'écrire sous une forme condensée les conditions auxquelles doit satisfaire le polynome f pour que Σ ait un contact d'ordre donné k avec la courbe C, ou bien avec la surface S elle-même. *M. Decuyper* (Lille).

Bell, P. O. Projective differential invariants of a curve of a surface. Duke Math. J. **18**, 697–705 (1951).

Soit S une surface analytique et C une courbe de S. Soit $ds^2=g_{\alpha\beta}du^\alpha du^\beta$ une forme quadratique invariante sur S par rapport à une transformation des paramètres. En utilisant les dérivées intrinsèques le long de C qui ont été définies dans l'article analysé ci-dessus, l'auteur met en évidence divers invariants parmi lesquels la courbure et la torsion projectives de C; de nouvelles caractérisations géométriques de ces éléments sont données. Puis sont introduits les tenseurs de courbure projective de S, une première forme fondamentale projective et la courbure totale projective. Ces quantités deviennent, lorsque le groupe projectif est remplacé par le sousgroupe des transformations orthogonales, les quantités métriques de mêmes noms. De même sont encore définis des R-centres de courbure principaux projectifs et un R-centre de courbure moyenne projectif qui peuvent se réduire aux points de mêmes noms de la géométrie métrique. *M. Decuyper* (Lille).

Terracini, Alessandro. Trasformazioni dualistiche di tipo nullo nel piano e sistemi (G) proiettivamente deformabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 89–94 (1951).

In projective non-homogeneous coordinates (x, y), consider the system Σ of ∞^1 curves: $T(x, y)=\text{const.}$, where T, not identically constant, is of class C^3 in a region of the projective plane. The equation $T(x, y)=\text{const.}$, defines a correspondence Γ, called a dual transformation of the null type, between a point A and a line a, which is tangent to the curve of Σ at A. In the present article, the theory of a non-special or general system Σ, in which the integral curves are not all straight lines, is developed.

If (A, a) and (B, b) are two pairs of corresponding elements under Γ, then as B approaches A in any manner, the limiting positions of the point of intersection of a and b and the line determined by A and B are the point G on a and the line g through A. The lineal element transformation T between the elements (A, g) and (G, a) induces a projectivity P_0 between the pencil of lines g through A and the range of points G on a, when A and a are fixed.

A principal conic of Σ is one through A and tangent to a such that G and g are always polar related with respect to it. At A, there are ∞^1 principal conics, all of which possess a common principal third order differential element E_3. As a principal conic varies along the curve of Σ through A, then two such consecutive conics at A intersect in three distinct characteristic points, none of which coincides with A. Under Γ, there are three distinct inflectional lines through A, none of which coincide with a, such that as A is approached along a second order differential inflectional element E_2 with one of these lines as tangent, there corresponds a second order differential cuspidal element E_2 with a cusp on a and a as cuspidal tangent. The three inflectional lines

through A and the three cusps on a correspond under P_0. At A, select a principal conic c_A whose three characteristic points are on the three inflectional lines through A. The totality of all such conics c_A over the region is termed a principal system of principal conics.

The above matters are related to systems of third order differential equations of the type (G) studied by Kasner [Differential-geometric aspects of dynamics, Amer. Math. Soc. Colloquium Publ., v. 3, New York, 1913], and Terracini [Univ. Nac. Tucumán. Revista A. 2, 245–329 (1941); these Rev. 4, 54]. For the special type (G_0) of the system (G) in which Σ is a related family, the conical satellite in the direction a through A of (G_0) is the conic c_A discussed above. Other interesting results in this direction are obtained.

J. De Cicco (Chicago, Ill.).

Terracini, Alessandro. Sistemi (G) proiettivamente deformabili di tipo speciale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 186–189 (1951).

This is a sequel to the paper reviewed above and extends the investigation to certain special systems (G) that require a different method of attack. *J. L. Vanderslice.*

Berezman, A. M. The canonical repère of a surface in four-dimensional projective space. Doklady Akad. Nauk SSSR (N.S.) 79, 373–376 (1951). (Russian)

Given a 2-dimensional surface in the space; at any point A_0 of it a frame is determined by the points A_i, $i=0, 1, \cdots, 4$, and an infinitesimal transformation of the frame is given by $dA_i=\omega_i^k A_k$. To obtain a canonical frame the author chooses $A_0A_1A_2$ to be the tangent plane to the surface, which implies $\omega^3=\omega^4=0$ so that ω_1^3, ω_2^3, ω_1^4, ω_2^4 are linear homogeneous in ω^1, ω^2. From this it follows that the rank of

$$\begin{vmatrix} a\beta-\alpha b & \frac{1}{2}(a\gamma-\alpha c) \\ \frac{1}{2}(a\gamma-\alpha c) & b\gamma-\beta c \end{vmatrix}$$

is invariant under the subgroup leaving $A_0A_1A_2$ tangent to the surface. One then considers the cases in which this rank is 2, 1 or 0. In the first case there are two possibilities of self-polar frames among which there is one which is self polar with respect to two hyperquadrics:

$$2x_0x_3=x_2^2-\tfrac{2}{3}x_1x_4=0 \quad \text{and} \quad 2x_0x_4-x_1^2-\tfrac{2}{3}x_2x_3=0.$$

If the rank is 1 the surface has one family of asymptotic lines $\omega^2=0$. If the rank is zero then the surface may be a plane, or a developable surface or it lies in three-space.

M. S. Knebelman (Pullman, Wash.).

Mishra, R. S. Union curves on the surface of reference of a rectilinear congruence. Math. Student 18 (1950), 37–41 (1951).

A union curve is a curve on the surface of reference of a rectilinear congruence which has the property that its osculating plane at each point of the curve contains the line of the congruence through that point. By straightforward calculations, expressions for the curvature and torsion of a union curve are found. *A. Fialkow.*

Mishra, R. S. Union curves and hyper-asymptotic curves on the surface of reference of a rectilinear congruence. Bull. Calcutta Math. Soc. 42, 213–216 (1950).

A hyper-asymptotic curve is a curve on the surface of reference of a rectilinear congruence which has the property that its rectifying plane at each point of the curve contains the line of the congruence through that point. Various geometric properties of hyper-asymptotic curves and union curves (defined in the preceding review) are derived. These relate to generalizations of the known theorems concerning asymptotic lines and geodesics on a surface.

A. Fialkow (Brooklyn, N. Y.).

Mishra, Ratan Shanker. Modification in Sannia's theory of line congruences and some deductions. Rev. Fac. Sci. Univ. Istanbul (A) 16, 95–101 (1951). (English. Turkish summary)

Sannia a substitué à la représentation analytique des congruences de droites par les deux formes de Kummer ($f=G_{\alpha\beta}du^\alpha du^\beta$, $\vartheta=\mu_{\alpha\beta}du^\alpha du^\beta$; $G_{\alpha\beta}=\lambda^i{}_{,\alpha}\lambda^i{}_{,\beta}$, $\mu_{\alpha\beta}=\kappa^i{}_{,\beta}\lambda^i{}_{,\alpha}$; $i=1, 2, 3$; $\alpha=1, 2$) une nouvelle représentation basée sur la considération des deux formes f et $\varphi=\xi_{\alpha\beta}du^\alpha du^\beta$, où $2\xi=E^{\alpha\beta}(G_{\alpha\gamma}\mu_{\beta\delta}+G_{\beta\gamma}\mu_{\alpha\delta})$, $E^{\alpha\beta}$ étant un tenseur contrevariant conjugué de $E_{\alpha\beta}=(\lambda^i{}_{,\alpha}\lambda^i{}_{,\beta}\lambda^j)$. L'auteur montre que dans certains cas il y a avantage à substituer au tenseur $\xi_{\alpha\beta}$ le tenseur non symétrique $\xi_{\alpha\beta}=E^{\gamma\delta}G_{\alpha\gamma}\mu_{\beta\delta}$. La condition pour que la surface de référence de la congruence soit la surface moyenne s'écrit alors: $\xi_{\alpha\beta}E^{\alpha\beta}=0$. De même, l'étude des surfaces réglées de la congruence dont les lignes de striction sont sur la surface de référence se présente sous une forme particulièrement simple, et conduit entre autres aux résultats suivants: (1) La somme des paramètres de distribution des deux surfaces précédentes issues d'un meme rayon de la congruence est égal au paramètre moyen de la congruence relatif à ce même rayon; (2) la différence des paramètres de distribution des deux surfaces réglées n'est jamais supérieure à la différence des paramètres principaux, et est nulle si la surface de référence est la surface moyenne. La forme simple de l'équation du réseau de la surface de référence déterminé sur celle-ci par les surfaces réglées dont elle contient les lignes de striction montre, très directement, que lorsque la congruence est formée des normales à la surface de référence, le réseau en question est le réseau asymptotique.

P. Vincensini (Marseille).

Finikov, S. A system of W-congruences with functional arbitrariness. Doklady Akad. Nauk SSSR (N.S.) 79, 197–199 (1951). (Russian)

Terracini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 4, 348–352 (1926)] considered so called systems of Bianchi, which are W-congruences depending on two parameters and with focal surfaces forming two one-parametric families. The author previously extended these studies [Izvestiya Akad. Nauk SSSR. Ser. Mat. 9, 79–112 (1945); these Rev. 7, 32; Učenye Zapiski Moskov. Gorod. Pedagog. Inst. 2, pp. 16ff. (1948)]. Here he proposes a W-system of which the congruences depend on two arbitrary functions of one variable and of which the focal surfaces form two families depending each on one arbitrary function of one variable. It is obtained by placing in $dA_i=\omega_i^k A_k$ ($i, k=1, 2, 3, 4$; two vertices A_1, A_2 of the tetrahedron A_i in the foci of a ray, two faces $A_1A_2A_3$, $A_1A_2A_4$ in focal planes):

$$\omega_1^4=0, \quad [\omega_1^3\omega_2^4+\omega_1^3\omega_3^4]=0,$$

which equations determine the focal surfaces (A_1). There must be an equation of the form

$$\omega_3^4=A\omega_1^3+B\omega_2^4+C\omega_1^3+D\omega_1^4.$$

Since $\omega_1^3=-B\omega_1^3+\lambda(\omega_2^4-C\omega_1^3)$, we obtain for the asymptotic lines $(A-BC-\lambda C^2)(\omega_1^3)^2+\lambda(\omega_2^4)^2=0$. The solution of these equations is further discussed. *D. J. Struik.*

*Takasu, Tsurusaburo. **Differentialgeometrien in den Kugelräumen. Band II. Laguerresche Differentialkugelgeometrie.** 2d printing. Taigado Publishing Co., Kyoto; Hafner Publishing Co., New York, N. Y., 1950. xxi+444 pp. $7.50.

The first edition, of which this is a photo-offset reprint, was published in 1939 by Maruzen Co., Tokyo, and reviewed in these Rev. 1, 86.

Clark, R. S. On the conformal theory of curves in a general differential metric space. Proc. London Math. Soc. (3) 1, 58–70 (1951).

The paper develops a conformal theory of curves in a general metric space. The work utilizes results of an earlier paper [same Proc. (2) 53, 294–309 (1951); these Rev. 13, 74] and is based on the idea of the conformal differential. The conformal derivative of a relative conformal tensor with respect to a curve C is defined. When applied to conformetric tensors, this differentiation process is a direct generalization of the one defined by the reviewer in connection with the conformal theory of curves in a Riemann space [Trans. Amer. Math. Soc. 51, 435–501 (1942); these Rev. 3, 307]. Conformal differentiation of a unit conformetric vector λ_i with respect to t defined along C leads to a system of Frenet equations which introduce conformally invariant curvatures. However, in these equations, neither λ_i nor t are defined in a conformally invariant manner. (The obvious choice, where λ_i is the tangent vector, is an unfruitful one since the conformal derivative of the tangent vector is zero.) The geometric interpretation of conformal derivative given by the reviewer is extended to apply to the case considered by the author. Some properties of conformal geodesics and paths are derived. *A. Fialkow.*

Stellmacher, Karl Ludwig. Geometrische Deutung konforminvarianter Eigenschaften des Riemannschen Raumes. Math. Ann. 123, 34–52 (1951).

Conformal properties of a Riemann space V_n are developed by considering the conformal geometry of V_n as a projective geometry of the $(n-1)$-dimensional isotropic hypersurface generated by all the geodesic null lines which emanate from a fixed point of V_n. The different conformal behavior of spaces with dimension numbers n less than 4 and at least 4 is related, in part, to the fact that geodesic null surfaces (generated by a 2-dimensional isotropic surface element) exist in V_n only if $n \geqq 4$. The conformal differences of spaces with odd and even dimension numbers corresponds to the known dissimilarity in the properties of isotropic surface elements in such spaces. Typical results are: (1) A p-dimensional totally isotropic manifold in a V_{2p} is necessarily flat. (2) If each totally isotropic surface, generated by the geodesic null lines of a totally isotropic surface element in a V_n ($n \geqq 4$) is a plane, then V_n is conformally euclidean. *A. Fialkow* (Brooklyn, N. Y.).

Eisenhart, Luther P. Generalized Riemann spaces. Proc. Nat. Acad. Sci. U. S. A. 37, 311–315 (1951).

A "generalized Riemann space" is an X_n with a not necessarily symmetric fundamental affinor g_{ij}. The symmetric part $g_{(ij)}$ is supposed to be of rank n and is used for raising and lowering of indices. Three connections are considered: $\{{}^h_{ij}\}$, $\Delta^h{}_{ij} = \frac{1}{2}g^{(hk)}(\partial_j g_{ik} + \partial_i g_{kj} + \partial_k g_{ij})$ and $\Gamma^h{}_{ij}$ defined by $\Gamma^h{}_{(ij)} = \{{}^h_{ij}\}$, $\Gamma^h{}_{[ij]} = \Delta^h{}_{[ij]}$ (differentiating index to the right). Some relations between the concomitants of these connections are discussed. *J. A. Schouten* (Epe).

Śiebodziński, W. Sur les espaces à parallélisme absolu doués d'une connexion semisymétrique. Colloquium Math. 2, 142–148 (1951).

If an n-space of distant parallelism is semi-symmetric, then its torsion tensor has the form $\delta_j{}^i\varphi_k - \delta_k{}^i\varphi_j$. The problem is to find all such spaces, given the φ_i. If p of the φ_i are independent, the solution depends on an arbitrary function of p variables. The solution is explicitly given.

J. L. Vanderslice (College Park, Md.).

Knebelman, M. S. Spaces of relative parallelism. Ann. of Math. (2) 53, 387–399 (1951).

In einer n-dimensionalen Mannigfaltigkeit bezeichne $C_P{}^Q$ einen stetigen Bogen mit dem Anfangspunkt P und Endpunkt Q. Jedem stetigen Bogen sei eine quadratische Matrix n-ter Ordnung $M(C_P{}^Q)$ zugeordnet, deren Elemente stetige Funktionen seien. $M(C_P{}^Q)$ genüge folgenden Forderungen: (a) M ist nicht singulär; (b) wenn C ein offener Bogen ist, dann ist $\lim_{Q\to P} M(C_P{}^Q) = I$ (Einheitsmatrix); (c) falls $C_P{}^R = C_Q{}^R + C_P{}^Q$ der P, Q, und R verbindende Bogen ist, dann sei $M(C_P{}^R) = M(C_Q{}^R)M(C_P{}^Q)$; (d) M ist im Fréchetschen Sinne differenzierbar. Eine Mannigfaltigkeit für die (a)–(d) gilt wird als eine solche mit relativen Parallelismus bezeichnet und M stellt ihre affine Strukturmatrix dar. Ist $\xi(x_0)$ ein kontravarianter Vektor in x_0, dann ist der längs $C^x{}_{x_0}$ parallelübertragene Vektor $\xi(C^x{}_{x_0})$ durch (1) $\xi(C^x{}_{x_0}) = M(C^x{}_{x_0})\xi(x_0)$ (Matrizenmultiplikation) definiert. Verf. ordnet nun der Strukturmatrix auf folgende Weise eine Matrix L zu, die einen affinen Zusammenhang bestimmt. Ist x_0 ein fester Punkt und $C^x{}_{x_0}$ ein beliebiger Bogen und $C_x{}^{x_1}$ der Bogen dessen Endpunkt $x+\alpha d_1 x$ ist, dann existiert wegen (d) der $\lim_{\alpha\to 0}[M(C^{x+\alpha d_1 x}_{x_0}) - M(C^x{}_{x_0})\,|\,\alpha]$ und kann auf die Gestalt $dM(C^x{}_{x_0}) = L(C(x))\cdot M(C^x{}_{x_0})$ gebracht werden. $L(C(x))$ hängt dabei nur von der lokalen Eigenschaften des Endpunktes x von $C^x{}_{x_0}$ ab, so dass in dieser Matrix ausser x nur die Differenziale dx, d^2x_2, $\cdots$ bis zu einer gewissen Ordnung eingehen. Die Matrizen M und L bestimmen einander umkehrbar eindeutig. Die Differentialgleichungen der Parallelverschiebung längs $C^x{}_{x_0}$ ist durch (2) $d\xi = L(C(x))\xi$ bestimmt. Mit Hilfe von (2) kann ein kovariantes Differenzial auf die übliche Weise definiert werden. Mit diesen Differentialoperator wird ein Torsionsskalar und eine Vektortorsion erklärt. Ist die Parallelübertragung unabhängig vom Bogen, dann ergeben sich die ebenen (affinen) Räume. Kovariante Ableitungen und Krümmungstensoren werden für den Fall ermittelt in dem $L(C(x))$ bloss von x und dx abhängt. Ist $\psi(C^x{}_{x_0})$ ein skalares Funktional für das $\psi(C^{x_0}_{x_0}) = 1$, dann heissen zwei Strukturmatrizen ${}^1M$ und M projektiv äquivalent falls ${}^1M = \psi M$ ist. Hieraus ergibt sich die projektive Strukturmatrix $\mathfrak{M} = {}^1M/(\det {}^1M)^{1/n} = M/(\det M)^{1/n}$. Der Projektivzusammenhang ist dann durch $\Pi = d\mathfrak{M}\cdot\mathfrak{M}$ bestimmt. Die projektive Theorie wird für den Fall dass Π nur Differenziale erster Ordnung enthält eingehender untersucht.

Zum Schluss dieser inhaltsreichen Arbeit werden von den Abbildungen des Raumes auf sich diejenigen untersucht, die die Parallelübertragung erhalten und gezeigt, dass sie eine Untergruppe der Kollineationsgruppe des Raumes bilden. Die Untergruppe fällt mit der Gruppe zusammen falls für den Raum der Torsionsskalar verschwindet.

O. Varga (Debrecen).

Géhéniau, J. Espaces de l'électromagnétisme. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 324–332 (1951).

After pointing out that in an electrostatic field potential may be used to define a metric along a line of force and

electric flux to define area on a surface, with similar results for a magnetic field, the author proceeds to discuss electromagnetic fields involving a pair of skew-symmetric tensors $H_{\alpha\beta}$, $K_{\alpha\beta}$ and a current vector C^α. These are subject to Maxwellian equations (1) $\partial_\beta H_{\overline{\alpha\beta}}=0$, (2) $\partial_\beta K_{\overline{\alpha\beta}}=C^\alpha$, where $\alpha\beta\overline{\alpha\beta}$ denotes an even permutation of 1234. The conditions (1) imply the existence of coordinates for which $H_{\alpha\beta}$ are constants in a finite domain of 4-dimensional space, on which account the space is said to be locally symplectic. It is stated that it is possible in general to reduce the forms $\Omega=H_{\alpha\beta}dx^\alpha\delta x^\beta$ (satisfying (1)) and $\overline{\Omega}=K_{\alpha\beta}dx^\alpha\delta x^\beta$ simultaneously to

$$\Omega=B(dX^1\delta X^2-dX^2\delta X^1)+h(dX^3\delta X^4-dX^4\delta X^3),$$
$$\overline{\Omega}=-b(dX^1\delta X^2-dX^2\delta X^1)+H(dX^3\delta X^4-dX^4\delta X^3),$$

where B and h are constants; then (2) give

$$\partial_2 H=C^1,\quad \partial_1 H=-C^2,\quad \partial_4 b=-C^3,\quad \partial_3 b=C^4.$$

The author discusses connections between $H_{\alpha\beta}$ and $K_{\alpha\beta}$ for coordinates independent of the field and the connection (3) $H_{\alpha\beta}=K_{\overline{\alpha\beta}}$ for special choice of coordinates. He shows that (3) is a special case of a more general connection of the form $H_{\alpha\beta}=pK_{\alpha\beta}+qK_{\overline{\alpha\beta}}$, which can be obtained by special choice of coordinates provided $H_{\alpha\beta}$ and $K_{\alpha\beta}$ satisfy an invariant relationship of the form $aH^2+2bHK+cK^2=0$, where $AB=\frac{1}{4}A_{\alpha\beta}B_{\overline{\alpha\beta}}$. This the author regards as a generalisation of results obtained by the reviewer [Proc. Symposia in Appl. Math. vol. 2, pp. 21–48, Amer. Math. Soc., New York, 1950; these Rev. 11, 401]. The connection is not clear to the reviewer, who was concerned with relations of the form $H_{23}=K_{14},\cdots,H_{14}=-K_{23},\cdots$, which differ from (3) in the matter of signs; also the author refers to $H^2-K^2=0$ as a sufficient condition, whereas the reviewer's condition was $H^2+K^2=0$ and was shown in Theorem IV to be insufficient as far as integrability is concerned. *J. L. Synge.*

Mogi, Isamu. A remark on recurrent curvature spaces. Kōdai Math. Sem. Rep. **1950**, 73–74 (1950).

This note is based on abstracts published in these Reviews, the author having had no access to original papers. Nor did he know of papers presented by A. G. Walker and the reviewer to the London Mathematical Society in 1947–1948, of which two [Walker, Proc. London Math. Soc. (2) **52**, 36–64 (1950); these Rev. **12**, 283; Ruse, paper reviewed below] have been published up to the time of the writing of this review. The note is concerned with properties, in the small, of Riemannian spaces K_n of recurrent curvature, that is, spaces for which the curvature tensor satisfies the relation $R_{\lambda\mu\nu\sigma,\omega}=\theta_\omega R_{\lambda\mu\nu\sigma}$ for some vector-field θ_ω. The author shows that, if K_n is an Einstein space of nonzero scalar curvature, then $\theta_\omega=0$ and K_n is symmetric in Cartan's sense. Assuming K_n not to be of this type, he deduces that it is locally decomposable into a product of two irreducible subspaces V_r, V_{n-r}. [This is not necessarily true, as is shown by Walker, loc. cit., theorem (10.1).] From this, and from an incorrect argument based upon it, he deduces that, if the Ricci tensor $R_{\lambda\sigma}$ is not identically zero, then K_n is symmetric. [That this is untrue follows from Walker, equation (58), or from the fact that the product of any V_2 and a flat V_{n-2} is a K_n.] He then obtains, correctly, the hitherto unknown identity $(R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2)\theta_\omega\theta^\omega=0$, from which it follows that either (i) $\theta_\omega\theta^\omega=0$, or (ii) the expression in brackets is zero. If the metric is positive definite, then (i) implies that K_n is symmetric, and (ii), coupled with the assumption $R_{\lambda\mu}=0$, implies that K_n is flat.

H. S. Ruse (Leeds).

Ruse, H. S. The Riemann complex in a four-dimensional space of recurrent curvature. Proc. London Math. Soc. (2) **53**, 13–31 (1951).

The definition of a K_n is known from an earlier paper [same Proc. (2) **50**, 438–446 (1949); these Rev. **10**, 571]. A K_n^* is either a K_n or a symmetric V_n in which there exists a vector field κ_λ, such that $R_{\mu\nu[\lambda}\kappa_{\rho]}=0$ [definition of Walker, ibid. **52**, 36–64 (1950); these Rev. **12**, 283]. Walker made an exhaustive analytic inquiry into the nature of these spaces. The methods developed in this paper for $n=4$ are more geometric. For $n=4$ the curvature affinor represents a quadratic complex in a projective three-dimensional space. But of the 56 types of quadratic complexes only two highly degenerate ones remain here. The complex consists either (α) of all lines meeting two distinct intersecting lines or (β) of all lines each taken twice meeting a (repeated) line. In the case (β) there exist two independent covariant constant vector fields (the "simple" case of Walker) and in the case (β) no two such fields exist (the "non-simple" case). This gives in all six subcases (flat space is excluded). Writing R_n for a flat V_n, using the sign $\times$ for the multiplication of spaces, and calling a space proper if it is non-product, the six cases are: (IA)(β) $K_4^*=V_2\times R_2$; κ_λ non-null; $R\neq 0$; two covariant constant (c.c.) non-null vector fields, perpendicular to one another and to κ_λ; (IB)(β) $K_4^*=K_3^*\times R_1$; κ_λ non-null; $R=0$; two c.c. vector fields one null and one non-null, perpendicular to one another and to κ_λ; (IIA)(β) $K_4^*=V_2\times R_2$; κ_λ null vector; $R\neq 0$; as in (IA); (IIB)(β) $K_4^*=K_3^*\times R_1$; κ_λ null vector; $R=0$, $R_{\mu\lambda}\neq 0$; as in (IB) but one of the fields in the direction of κ_λ; (IIC)(β) proper; κ_λ null vector; $R_{\mu\lambda}=0$; two c.c. vector fields, both null and perpendicular to one another; (IID1)(α) Einstein space $R=0$; κ_λ null vector; one c.c. vector field with the direction of κ_λ; $K_{\rho\lambda\lambda\sigma}={}^0R_{\rho\lambda\lambda\sigma}\equiv\frac{1}{4}I_{\rho\mu\rho\sigma}I_{\lambda\sigma\tau\omega}R^{\rho\sigma\tau\omega}$; (IID2)($\alpha$) conformal flat; κ_λ null vector; as in (IID1); $K_{\rho\lambda\lambda\sigma}=-{}^0R_{\rho\lambda\lambda\sigma}$.

J. A. Schouten (Epe).

Ruse, H. S. A classification of K^*-spaces. Proc. London Math. Soc. (2) **53**, 212–229 (1951).

This is classification of K_n^* for general values of n. A k_3^* is a space with the metric $ds^2=\psi(x,z)dx^2+2dxdy+dz^2$ and a k_4^* has the metric $ds^2=\psi(x,z)dx^2+2dxdy+2dzdt$. There are six cases: (IA)(β) $K_n^*=V_2\times R_{n-2}$; κ_λ non-null; $R\neq 0$; $n-2$ c.c. (covariant constant) non-null fields, perpendicular to one another and to κ_λ; (IB)(β) $K_n^*=k_3^*\times R_{n-3}$; κ_λ non-null; $R=0$; $n-3$ c.c. non-null fields and one c.c. null field, all perpendicular to one another and to κ_λ; (IIA)(β) $K_n^*=V_2\times R_{n-2}$; κ_λ null vector; $R\neq 0$; as in (IA); (IIB)(β) $K_n^*=k_3^*\times R_{n-3}$; κ_λ null vector; $R=0$, $R_{\mu\lambda}=0$; $n-3$ c.c. non-null fields perpendicular to one another and to κ_λ and one c.c. null field in the direction of κ_λ. (IIC)(α) $K_n^*=k_4^*\times R_{n-4}$; κ_λ null vector; $R_{\mu\lambda}=0$; $n-4$ c.c. non-null fields and two c.c. null fields all perpendicular to one another. (IID)(α) K_n^* non-simple; Walker [same Proc. (2) **52**, 36–64 (1950); these Rev. **12**, 283] has given a canonical form.

J. A. Schouten (Epe).

Laptev, G. F. On fields of geometric objects on imbedded manifolds. Doklady Akad. Nauk SSSR (N.S.) **78**, 197–200 (1951). (Russian)

In the present paper the author confines himself to fields of geometrical objects in manifolds with finite continuous fundamental groups. If G is this group and its basic invariant forms are $\omega^{s_{12}}$, the equations of structure are then

$$D\omega^{s_{12}}=\tfrac{1}{2}C^{s_{12}}_{s_{23}s_{13}}[\omega^{s_{23}}\omega^{s_{13}}],\quad s_{12}=1,\cdots,r.$$

The r-parameter group G with a subgroup H not containing a normal subgroup of G is called an abstract geometrical element G (with fundamental group G and supporting groups H). The linearly independent invariant forms whose vanishing determines H are taken as basic forms ω^{s_1}, $s_1=1, \cdots, \mathfrak{r}$—these are the primary forms—and the remaining invariant forms of G are denoted by ω^{s_2}, $s_2=\mathfrak{r}+1, \cdots, r$ and are called the secondary forms. A point $X^1, \cdots, X^N$ of any representation space of H of the element GH is called a geometrical object of this element. These components must satisfy the completely integrable Pfaffian system $\omega^{s_1}=0$, $dX^J=\Xi^J(_{p_2}X)\omega^{p_2}$. Tensors are characterized by such a system with Ξ being linear forms (with constant coefficients) $\Xi^J{}_{L p_2}X^L$. The more general geometric objects are obtained by considering submanifolds of the group manifold of H defined by

$$\omega^{\bar{s}_1}=\Lambda^{\bar{s}_1}_{\hat{p}_1}\omega^{\hat{p}_1}, \quad \bar{s}_1=1, \cdots, n, \quad \hat{p}_1=n+1, \cdots, \mathfrak{r}$$

and by the process of extending this group to some order ν. The paper contains one or two bothersome misprints and a set of functions that are not clearly defined.

M. S. Knebelman (Pullman, Wash.).

Vagner, V. V. The theory of composite manifolds. Trudy Sem. Vektor. Tenzor. Analizu. **8**, 11–72 (1950). (Russian)

The author bases his work on the theory of Veblen and Whitehead's "Foundations of geometry", to the translation of which he wrote a supplement. The result is a generalized theory of differential geometrical objects. If B_n is some space of Veblen-Whitehead of class r, then to every point P of B_n is associated a local space $L_m(P)$ of class u, the local space. The manifold of all local space L_m is called a composite manifold and written $L_m(B_n)$. The first chapter treats especially $L_m(B_1)$ and $L_m(B_2)$; in it we also find the necessary and sufficient condition for the existence of a local holonomic linear connection in a $L_m(B_n)$. The second chapter deals with the isomorphy of connections in $L_m(B_n)$ with local Klein spaces, the third with so called differential connections of order v in an X_n, where the fundamental group for the tangent spaces T_{vn} of order v is the differentiable group $\mathfrak{D}_{N_{vn}}$ [see Doklady Akad. Nauk SSSR (N.S.) **69**, 293–296 (1949); these Rev. **11**, 461]. Then follow derivatives of Lie in the sense of Schouten-Van Kampen and the absolute basic differentiation of a differential object. At the conclusion we find a list of twelve papers of the author on this and related subjects. *D. J. Struik* (Cambridge, Mass.).

Vagner, V. V. The geometry of the generalized Cartan spaces and the theory of geometric differential objects. Doklady Akad. Nauk SSSR (N.S.) **77**, 777–780 (1951). (Russian)

The geometry of an arbitrary Klein space reduces to the geometry of a manifold X_n with given characteristic geometric object of which the pseudogroup of invariance reduces to a Lie group of point transformations of the X_n in itself. (Reference is made to an essay by the author entitled "Supplement to the book of O. Veblen and J. Whitehead, Foundations of geometry", 1949.) Cartan generalized Klein spaces into manifolds with non-holonomic connections corresponding to an arbitrary given group of Lie. The present paper relates these Cartan spaces to the author's theory, but, partly because the notation is not always well explained, it is hard to follow his reasoning. Starting point is an X_n in an $X_{n+(m)}$ with connection $\delta\eta^a=d\eta^a+\Gamma_s{}^a(\xi^b, \eta^t)d\xi^s=0$ and a field of local points given along the curve $\xi^s=\xi^s(t)$ of the basic X_n with equation $\eta^a=\eta^a(t)$. A mapping is considered of the local spaces $X_m(\bar{t})$ associated with the points on the curve belonging to some neighborhood of the point (t) on the local space $X_m(t)$ associated with the point. If ${}^s\eta^a={}^s\eta^a(\bar{t})$ represents the image of the points under these mappings, then the vector $\delta^s\eta^a/dt^s=(d^s{}^s\eta^a/d\bar{t}^s)_{\bar{t}=t}$ in the tangent space t $(s=1, \cdots, v)$ of order v associated with the local points of $\eta^a(t)$ is called the vector of deviation of order v of the given field of local points at the point t. This vector is basic for the further definition of the geometrical objects. Several special cases are studied, especially $n=m$. Reference is also made to the author's paper reviewed above.

D. J. Struik (Cambridge, Mass.).

Nijenhuis, Albert. X_{n-1}-forming sets of eigenvectors. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 200–212 (1951).

The necessary and sufficient conditions for the eigenvectors of a tensor field of valence two in V_n with distinct eigenvalues to be V_{n-1}-normal were given by Schouten [Der Ricci-kalkül, Springer, Berlin, 1924, p. 196] in a form containing the eigenvectors. Tonolo [Ann. Mat. Pura Appl. (4) **29**, 99–114 (1949); Pont. Acad. Sci. Acta **13**, 29–53 (1949); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 438–444 (1949); these Rev. **12**, 54; **11**, 461] gave for $n=3$ a form containing neither the eigenvectors nor the eigenvalues. Schouten [see the paper reviewed below] gave the generalization for V_n and an abbreviation of Tonolo's proof. The first section of this paper is a review of these latter investigations. In the second section the metric is eliminated, that is, the problem is considered whether an affinor field $h_\lambda^{\cdot s}$ in S_n with distinct eigenvalues has X_{n-1}-forming covariant eigenvectors. Several forms of necessary and sufficient conditions are given. They contain a new differential comitant

$$H_{\mu\lambda}^{\cdot\cdot s}=2h_{[\mu}^{\cdot\cdot\rho}\partial_{|\rho|}h_{\lambda]}^{\cdot s}-2h_\rho^{\cdot\cdot s}\partial_{[\mu}h_{\lambda]}^{\cdot\rho}$$

of $h_\lambda^{\cdot s}$. The comitant can also be formed with respect to two different fields $h_{\lambda}^{\cdot s}$ and $h_{\lambda}^{\cdot s}$. It is closely connected with the Lie derivatives with respect to the fields of eigenvectors. The meaning of $H_{\mu\lambda}^{\cdot\cdot s}=0$ is investigated. Using these differential comitants several forms of the necessary and sufficient conditions for the metric case can be given. The last section contains some geometrical considerations on the Lie derivative, the object of anholonomity $\Omega_{\mu\lambda}^{\cdot\cdot s}$ and $H_{\mu\lambda}^{\cdot\cdot s}$.

J. A. Schouten (Epe).

***Schouten, J. A. Sur les tenseurs de V_n aux directions principales V_{n-1}-normales.** Colloque de Géométrie Différentielle, Louvain, 1951, pp. 67–70. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

In 1949, A. Tonolo succeeded in giving conditions for the principal directions of a tensor field $h_{\mu\lambda}$ in V_3 or R_3 to be V_2-forming in such a form that they could be verified without solving the characteristic equation of the tensor [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fiz. Mat. Nat. (8) **6**, 438–444 (1949); Pont. Acad. Sci. Acta **13**, 29–53 (1949); Ann. Mat. Pura Appl. (4) **29**, 99–114 (1949); these Rev. **11**, 461; **12**, 54]. The author generalises this result to the case when in V_n the eigenvectors are V_{n-1}-forming, and gives a simplified derivation of the necessary and sufficient conditions; they have almost the same form as those of Tonolo [Pont. Acad. Sci. Acta **13**, 29–53 (1949); these Rev. **11**, 461], and are derived under the assumption that the characteristic roots of $h_{\mu\lambda}$ are all distinct and different from zero.

In a note at the end, announcement is made of some results obtained by the reviewer concerning when the covariant eigenvectors of a mixed affinor field h_λ^κ (with distinct eigenvalues, not necessarily different from zero) in X_n are X_{n-1}-forming [see the preceding review]. There is no metric or linear connection in that case, and the question appears to depend on a differential concomitant of h_λ^κ alone.

A. Nijenhuis (Amsterdam).

Kilmister, C. W. Tensor identities in wave-tensor calculus. Proc. Roy. Soc. London. Ser. A. **207**, 402–415 (1951).

In the first part of this paper the author determines the relations that must be satisfied by the tensor set in S_4 corresponding to a linear function (two-index four-component spinor) in a (spin) space V_4 in order that the linear function be idempotent. The second part of the paper is concerned with a classification of linear functionals (four-index four-component spinors) in terms of symmetry properties on the indices. *A. H. Taub* (Urbana, Ill.).

Chaki, M. C. Some formulas in tensor calculus. Bull. Calcutta Math. Soc. **42**, 249–252 (1950).

The methods indicated by Sen [same Bull. **42**, 1–13, 177–187 (1950); these Rev. **12**, 205, 533] are utilized to develop formulas concerning an arbitrary affine connection and an arbitrary tensor. *A. Fialkow* (Brooklyn, N. Y.).

NUMERICAL AND GRAPHICAL METHODS

***Neĭšuler, L. Ya. Tablicy perevoda pryamougol'nyh dekartovyh koordinat v polyarnye. [Tables for the Transformation of Rectangular Cartesian into Polar Coordinates].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 291 pp.

The rectangular coordinates (x, y) correspond to the polar coordinates (s, α), if $s=(x^2+y^2)^{\frac{1}{2}}$, $\alpha=\tan^{-1}(y/x)$. This table is for x and y integers up to 10,000. It is generally supposed that $y \geqq x$. If x is the greater, then instead of α, $90°-\alpha$ is found, the x now being regarded as y of the table.

The extreme left-hand column of each of the pages 7–290 gives values of y. The first range of values of y (pages 7 to the first column of page 12) is 1000–1090. Corresponding to these values are ranges of x: 0–10 (5), 10–20 (15), $\cdots$, 400–420 (410), $\cdots$, 820–860 (840), $\cdots$, 1070–1105 (1105). Under each of these ranges of x are columns giving values of s, α, Δ. The values of s and α given correspond exactly to the bracketed values 5, 15, $\cdots$, 1105, which are almost invariably the means of values at the ends of the ranges; an exception is illustrated by 1105. The values of Δs and $\Delta\alpha$ given in the Δ column are for interpolating so as to get the values corresponding to the exact x in a given problem. (In the latter part of the volume the Δ tables sometimes run up to 22 entries.) The angle α is given to a tenth of a minute, and s to 5S or 6S, usually 1D. For such a pair of values as 4 and 3 one would turn to $y=4000$ $x=3000$. Similarly for any other y less than 1000.

Illustrative values are worked out on pages 4–6. This table is far more elaborate than anything previously published for this particular purpose. *R. C. Archibald.*

Petschacher, Martha. Tabelle di funzioni ipergeometriche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 389–420 (1950) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **297** (1951).

The hypergeometric series $F(x)=F(a, b; c; x)$ with

$$a+b=\mu+\frac{1}{k-1}, \quad ab=-\frac{\mu(\mu-1)}{2(k-1)}, \quad c=1+\mu$$

and the related functions $Y(x)=x^{1-c}F(x)/\Gamma(1+\mu)$ arise as solutions of Chaplygin's differential equation in compressible flow theory. In the present paper numerical tables of $F(x)$ and $Y(x)$ are provided for $k=1.4$, $\mu=.125(.125).875$, $1.25(.25)2(.5)7(1)10$, $x=0(.02)1$. For related tables see Ferguson and Lighthill [Proc. Roy. Soc. London. Ser. A. **192**, 135–142 (1947); these Rev. **9**, 351] and Huckel [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1716 (1948); these Rev. **10**, 329]. *A. Erdélyi* (Pasadena, Calif.).

Kulikov, D. K. Formulas and tables for the differential correction of parabolic orbits. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. **4**, 451–487 (1951). (Russian)

Let $10H=8+(\lambda-20)/(2+\lambda^{\frac{1}{2}})$, $10K=-3+4\lambda/(2+\lambda^{\frac{1}{2}})$,

$$b=\tfrac{1}{2}(n^2-n+\tfrac{1}{6}), \quad b_1=\tfrac{1}{12}(2n^3-3n^2+2n-\tfrac{1}{2}),$$
$$b_2=\tfrac{1}{24}(n^4-2n^3+n-\tfrac{1}{10}),$$
$$b_3=\tfrac{1}{720}(6n^5-15n^4+10n^3-4n+\tfrac{3}{2}),$$
$$b_1'=\tfrac{1}{2}(n^2-n+\tfrac{1}{3}), \quad b_2'=\tfrac{1}{12}(2n^3-3n^2+\tfrac{1}{2}),$$
$$b_3'=\tfrac{1}{24}(n^4-2n^3+n^2-\tfrac{3}{15}).$$

Tables are given for H and K for

$$\lambda=1(0.01)10(0.1)100(1)1000; \text{ 6D}.$$

Tables for the b's and b''s are given for $n=0(0.01)0.50$; 6D for b, b_1, b_1'; 5D for b_2, b_2', b_3'; 4D for b_3.

Fröberg, Carl-Erik. On determination of proton-proton interaction from scattering experiments. Ark. Fys. **3**, 1–34 (1951).

The author undertakes to determine an unknown central nuclear interaction in addition to the Coulomb force by means of scattering data. Mathematically this requires finding an unknown potential $V(r)$ in the radial wave equation which will account for the observed phase δ. This problem in turn is made to depend on two independent solutions of the equation

$$d^2y/d\rho^2+(1-2\alpha/\rho)y=0.$$

One of these solutions, the regular Coulomb wave function u, was computed by the Mathematical Tables Project (unpublished). The present paper provides tables for the other solution, the irregular Coulomb wave function v. The values of v and v' are tabulated to six decimal places for nineteen values of α from 0.01 to 1.00. The range for ρ varies from 0–6, to 0–1, depending on α, and the step-interval for ρ also varies with α, being sometimes as small as 0.002, sometimes as large as 0.05. For small ρ the values were found from series, for large ρ from the asymptotic form, and for intermediate values by numerical integration on the Swedish automatic relay computing machine BARK.

W. E. Milne (Corvallis, Ore.).

***Krylov, A. N. Lekcii o priblizennyh vyčisleniyah. [Lectures on Approximate Calculations].** 5th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 398 pp.

The first edition of this book appeared in 1911, the second in 1932, the third in 1935, and the fourth as vol. 3, part 1, of his collected works. There have been only minor changes since the 3d edition.

Hammer, Preston C. Best starting values for an iterative process of taking roots. United States Atomic Energy Commission Rep. **AECU-708**, ii+8 pp. (1950).

This paper describes a method for the choice of the best starting values for one type of iteration for square roots when one is concerned with limits on absolute or relative errors for a minimum fixed number of iterations. Higher order roots are also considered. *E. Frank.*

★**Hammer, Preston C. Best starting values for an iterative process of taking roots.** Proceedings, Computation Seminar, December 1949, pp. 132–134. International Business Machines Corp., New York, N. Y., 1951.

The content of this paper is the same as that of the paper reviewed above. *E. Frank* (Chicago, Ill.).

Olver, F. W. J. A further method for the evaluation of zeros of Bessel functions and some new asymptotic expansions for zeros of functions of large order. Proc. Cambridge Philos. Soc. **47**, 699–712 (1951).

Let $\rho=\rho(n, t)$ be a zero of

$$\mathcal{C}_n(z)=J_n(z)\cos\pi t-Y_n(z)\sin\pi t.$$

The author has shown [Proc. Cambridge Philos. Soc. **46**, 570–580 (1950); these Rev. **12**, 288] that integrating a (third-order non-linear) differential equation for $\rho=\rho(t)$, n being fixed, is a convenient way of determining the larger zeros of $\mathcal{C}_n(z)$. He now presents a convenient way of determining, to high accuracy (say 10D), the smaller zeros for not too small values of n (e.g. $n=0(\frac{1}{4})20$). The method is the integration of the equation:

$$\partial\rho/\partial n=2\rho\int_0^\infty K_0(2\rho\sinh\tau)\exp(-2n\tau)\,d\tau,$$

after determining an asymptotic expansion of the integral. Numerical values of $\mathcal{C}_n'(\rho)$, of the zeros σ of $\mathcal{C}_n'$ and of the stationary values $\mathcal{C}_n(\sigma)$, can be obtained by similar considerations. The second part of the paper is devoted to the determination of the asymptotic expansion (for large n) of the same four quantities (corresponding to the first five zeros and turning values). The first few coefficients in the expansions are determined explicitly and also numerically. This work is supporting the program of the Royal Society Mathematical Tables Committee for the evaluation of the zeros of J_n, Y_n, J_n', Y_n' for n an integer, or half an odd integer. *J. Todd* (Washington, D. C.).

Hestenes, Magnus R., and Karush, William. A method of gradients for the calculation of the characteristic roots and vectors of a real symmetric matrix. J. Research Nat. Bur. Standards **47**, 45–61 (1951).

Let $\lambda_1<\lambda_2<\cdots<\lambda_r$ be the distinct eigenvalues of the real symmetric matrix A, and let y_k denote any eigenvector belonging to λ_k. For a vector x let $\mu(x)=(Ax, x)/(x, x)$, and let $\xi(x)=Ax-\mu(x)x$, a vector proportional to the gradient of $\mu(x)$. In the general method of gradients one selects x^0 and a sequence of scalars γ^i, and defines $x^{i+1}=x^i-\gamma^i\xi(x^i)$ $(i=0, 1, 2, \cdots)$. Conditions are given on γ^i so that: (1) $x^i\to Ly_1$ (L=scalar), or that (2) $x^i/|x^i|\to y_1/|y_1|$; either (1) or (2) implies that $\mu(x^i)\to\lambda_1$. (Changing the sign of the γ^i yields y_r and λ_r.) Two special methods are suggested for automatic computing machinery and are studied in detail. In the first, the "method of constant α," $\gamma^i=\alpha=\beta/(\lambda_r-\lambda_1)$; if $0<\beta<2$, (1) holds and, if $0<\beta<1$, one also has (3) $\xi(x^i)\to Ly_2$. In the second, the "method of optimal α," γ^i is the real number γ minimizing $\mu(x^i-\gamma\xi(x^i))$; here (1) always holds, while (3) never does. It is shown that all the vectors $y_1, \cdots, y_r$ can theoretically be obtained from one sequence $\{x^i\}$ by a method similar to one of Lanczos [same J. **45**, 255–282 (1950); these Rev. **13**, 163]. The rate of linear convergence of these methods is studied. The case where x^0 is orthogonal to certain invariant spaces of A is treated precisely. *G. E. Forsythe* (Los Angeles, Calif.).

Masuyama, Motosaburo. On a numerical method of solution of the equation $|K-M\theta|=0$. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, 26–28 (1951).

Let the dominant root θ_m of the determinantal equation (*) $|K-M|\theta=0$ be assumed unique, where K, M are real symmetric matrices with $|M|\neq 0$. Let $T=M^{-1}K$, and let $x_n=T^nx_0$, where x_0 is an arbitrary vector. Define the three scalars $C_n=x_n'\cdot M\cdot x_n$, $L_n=x_n'\cdot K\cdot x_n$, $Q_n=x_n'\cdot K\cdot T\cdot x_n$. The ratio $\theta_R=Q_n/L_n$ is recommended as an approximation to θ_m; no reference is made to similar proposals (at least when $M=I$) of Aitken [Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)] and others. The author shows that commonly, for sufficiently large n, (**) $0\leqq\theta_m-\theta_R\leqq\theta_m-\theta_P$, where θ_P is the ratio of the jth coordinates of $T\cdot x_n$ and x_n. The proof assumes that T has only linear elementary divisors, and it is erroneously stated that (**) always holds for sufficiently large n. Some upper bounds for θ_m are given, one under the tacit hypothesis that all roots θ_i of (*) are real. When all $\theta_i\geqq 0$ it is asserted that $\theta_m\leqq L_n/C_n+D_M$, where $D_M=(L_n/C_n)(Q_n/L_n-L_n/C_n)/(L_n/C_n+Q_n/L_n-\text{trace } T)$.

G. E. Forsythe (Los Angeles, Calif.).

★**Dwyer, Paul S. Linear Computations.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. xi+344 pp. $6.50.

For the most part this is a collection of pivotal methods for solving systems of linear equations and closely related problems on desk calculators. Some other chapters deal with designing and efficiently carrying out computations with approximate numbers.

In chapter 1 the author describes various features of desk computing machines. Chapter 2 deals with four systems of recording the accuracy of an approximate number x. Since independent error estimates can be made for the linear computations of this book, "incomplete numbers" (numbers without tolerances) are generally used. In chapter 3 principles of computational design are set forth, with the purposes of controlling blunders and round-off errors, of providing intermediate and final checks, and of minimizing computing and recording time. Synthetic methods and indirect devices to defer approximations in the evaluation of formulas are illustrated.

With chapter 4 the treatise on linear computations properly begins. In eight chapters we are given the basic Gaussian pivotal elimination with seemingly endless variations, all expressed in expanded notation without matrices. Explicit directions are given for every method, usually with a table illustrating the arrangement of the computing sheet and one or more tables illustrating the computation for a numerical system of order 3, 4, or 5. We learn of the exact method of multiplication and division with or without (exact) division, and of the approximate methods of row division, diagonal division, and single division, with variations. The compact methods of single division include those of Doolittle, Banachiewicz, Crout, and others, which the author considers to be the best arrangements of elimination for systems with unsymmetrical matrices. For systems with positive definite symmetrical matrices the author seems to

prefer the square-root method of Cholesky. The author recapitulates the methods by giving relations between the coefficients appearing in them. We are then told how to solve several of the related systems formed by suppression of certain variables and equations or by addition of columns to the right-hand side. After a chapter introducing determinants there is a long one on evaluating them and their principal minors. Besides the pivotal methods of the earlier chapters there is a sketch of Dodgson's non-pivotal condensation. A section deals with the accuracy of a determinant whose elements are approximate numbers. A chapter tells how to evaluate a linear form $a_1x_1+\cdots+a_ix_i$ when the x's are given implicitly as the solutions of a system of linear equations. Now matrices are introduced for the first time, with mention of column-by-column and row-by-row, as well as the usual multiplication rules. (But there is no mention that only the latter is associative.) The more efficient of the elimination methods are then applied to the calculations of inverse matrices, with three variations. A chapter discusses useful bordering devices helpful in either solving equations or inverting matrices.

The errors in linear computations due to variations in the coefficients are discussed in chapter 17. Approximate absolute upper bounds are given for the error of each component; the author states (p. 288) that probabilistic bounds are less satisfactory in principle and are, moreover, not much tighter anyway. No bounds are given for the length of the vector error. In chapter 15 are given Leverrier's and Frame's methods for determining the characteristic equation of a matrix. It is good to see Frame's method in print at last but, in view of the author's preference for efficient elimination methods for solving linear equations, one would expect to see the elimination methods of Krylov, Danilevski, and Samuelson, which require many fewer multiplications. [For references on all these method except Frame's, see H. Wayland, Quart Appl. Math. **2**, 277–306 (1945); these Rev. **6**, 218.] At the end of chapter 16 on extension (=partitioning) methods, iterative methods for solving linear equations are dismissed with a few references. "The author feels that linear problems that should be undertaken with a hand machine are, for the most part, problems for which direct pivotal methods are satisfactory." (p. 252)

A chapter mentions ways of avoiding early approximations in evaluating certain statistics, to reduce error accumulations. In the final chapter, after discussing methods of interpolation, the author expresses his expectation of an increased use of desk machinery for linear problems. He believes that the methods of the book are feasible for up to 15 or 20 variables (although his examples have never more than 5), and that linear problems involving more variables would require IBM or other special equipment, but that the majority of such problems (at least those "based on experimental evidence") should be reduced to desk size before computation begins.

The book has over 160 carefully prepared tables, mostly numerical illustrations of specific elimination methods. We find many exercises for the reader. There are extensive references for each chapter; these cover the literature in English exceptionally well, but are weaker in French and German titles and omit Russian work entirely. The book should be a standard reference for matrix elimination methods, in "supplying definite techniques for specific problems" (p. vi). Its numerous tables, great detail, and elementary notation should appeal to a wide group of computers. Because of its emphasis on techniques to the neglect of tools, the book does not appeal to the reviewer as a college textbook. The student prepared to devote this much attention to linear problems will advance much further by first learning standard elementary matrix theory, with emphasis on the triangular decomposition and matrix norms.

Minor criticisms: On p. 18 the purpose of range numbers seems to be defeated by rounding the upper bound 418.94397 down to 418.9 instead of up to 419.0. On p. 252 the use of subscripts for both components and vectors is confusing. On p. 262 without warning the symbol $|c|$ is used for a matrix with elements $|c_{ij}|$. *G. E. Forsythe.*

Couffignal, Louis. Sur la résolution numérique des systèmes d'équations linéaires. II. Revue Sci. **89**, 3–10 (1951).

In an earlier paper [Revue Sci. **82**, 67–78 (1944); these Rev. **8**, 128] the author explained the mechanics of solving systems of linear algebraic equations. In this paper he investigates the accuracy of the solution, distinguishing between errors due to inaccuracies in the values of the coefficients in the given equations (which he calls "physical errors") and those errors which accumulate during the process of solution due to roundoff (called "calculational errors"). The physical error is studied and an approximate analytical expression for it is given. The calculational errors are examined in much detail and a list of practical rules is given for keeping the calculational errors as small as possible. As in the first paper the author avoids the term "matrix" but introduces various terms such as "moyenne", "marge", "erreur de chute", etc., with meanings which were strange to the present reviewer. *W. E. Milne.*

Šura-Bura, M. R. Estimation of errors in the computation of inverses of matrices of high order. Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 121–150 (1951). (Russian)

An exposition of the results of a paper by von Neumann and Goldstine [Bull. Amer. Math. Soc. **53**, 1021–1099 (1947); these Rev. **9**, 471].

Opler, Ascher. Monte Carlo matrix calculation with punched card machines. Math. Tables and Other Aids to Computation **5**, 115–120 (1951).

The von Neumann-Ulam Monte Carlo procedure for matrix inversion, described by Leibler and the reviewer [same journal **4**, 127–129 (1950); these Rev. **12**, 361], is adapted to an IBM 405 tabulator for matrices of orders up to 10, depending on the number of available selectors. The procedure is here limited to the inversion of $I-A$, where $\sum_j|a_{ij}|<1$ for each i. A deck is prepared for the matrix A, with a relative frequency of j punches in the ith card columns equal to $|a_{ij}|$, and with certain other punches. After one pass of the deck the tabulator prints total draws, total games, and net scores, which are readily convertible into one row of the approximate inverse matrix. The author says that the working procedure is far simpler than for any previously reported inversion procedure, and that the number of operations grows like n^2. (For elimination procedures the number grows like n^3.) Three examples are given for $n=7$. Wiring directions are given. The author says the IBM 101 statistical machine can handle n up to 30. If $\sum_j a_{ij}=1$, a modified procedure calculates a matrix power A^r. *G. E. Forsythe* (Los Angeles, Calif.).

Banachiewicz, Tadeusz. Sur la résolution des équations normales de la méthode des moindres carrés. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 41 (1948), 63–68 (1950). (French. Polish summary)

(This review uses matrix notation, with T for transpose, whereas the author uses cracovians.) In the square root method of solving a system (*) $Ax+L=0$, where A is a positive definite matrix, after factoring A into a product $A=a^Ta$, where a is triangular, one computes $z=(a^T)^{-1}L$ and then $x=-a^{-1}z$. The author considers this as requiring three "stages," and proposes the following trick to reduce the number of stages to two. Convert (*) into a singular homogeneous system (**) $By=0$ of order $n+1$ in the column $y=(x^T, 1)^T$ by adding the equation $Lx+c=0$. In solving (**) by the square root method one gets the triangular decomposition $B=b^Tb$ without having to know the number c. It is then necessary only to solve the singular system $by=0$ with the last component normalized to 1. There is no attempt to show that the trick saves operations in using condensed arrangements of the square root method.

G. E. Forsythe (Los Angeles, Calif.).

Zuhovickiĭ, S. I. An algorithm for the solution of the Čebyšev approximation problem in the case of a finite system of incompatible linear equations. Doklady Akad. Nauk SSSR (N.S.) 79, 561–564 (1951). (Russian)

The author describes a method of successive approximations for computing the best solution of the incompatible linear equations

$$\sum_{j=1}^{n} a_{ij}\xi_j = b_i \qquad (i=1, 2, \cdots, m),$$

where $m>n$. The best solution is to be taken in the sense of Čebyšev, i.e., the set of ξ's which gives the least value for $\max_i|\sum a_{ij}\xi_j-b_i|$. *W. E. Milne* (Corvallis, Ore.).

Reiz, Anders. Quadrature formulae for the numerical calculation of mean intensities and fluxes in a stellar atmosphere. Ark. Astronomi 1, 147–153 (1950).

The author derives an improved formula for calculating the coefficients p_i in quadrature formulas of Gaussian type

$$\int_a^b f(x)w(x)dx = \sum_{i=1}^{n} p_i f(x_i)$$

where the p_i depend only on $w(x)$. The method is then applied to the particular set of weight functions $w(x)=\int_1^\infty e^{-tx}t^{-n}dt$ and tables of the p_i and x_i are furnished. Application is also made to the Laguerre case $w(x)=e^{-x}$. *W. E. Milne.*

Tortorici, Paolo. Resti nelle formule di cubatura. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 287, 8 pp. (1950).

The author derives a formula of cubature analogous to the familiar trapezoidal rule for quadrature and obtains two formulas for the error, the first composed of two simple integrals and one double integral, the second expressible in terms of values of the three second order partial derivatives of the integrand. *W. E. Milne* (Corvallis, Ore.).

Fempl, Stanimir. Näherungsformel zur Mantelberechnung des schiefen Kreiskegels. Srpska Akad. Nauka. Zbornik Radova, Knj. 7, Matematički Institut, Knj. 1, 135–142 (1951). (Serbo-Croatian. German summary)

***Collatz, Lothar. Numerische Behandlung von Differentialgleichungen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Band LX. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1951. xiii+458 pp. (1 plate). 45 DM; Bound, 48 DM.

This text contains the most thorough and comprehensive presentation of numerical methods for differential equations that has yet been published. The term "numerical methods" is liberally interpreted, and far from being content with a mere description of procedures the author gives a detailed theoretical analysis of each process described. The crucial question of error control has received the careful attention that its importance merits.

A quick glance through the 458 pages discovers the major headings: (I) initial-value problems for ordinary equations; (II) boundary-value (i.e., two end-point) problems for ordinary equations; (III) initial-value problems and initial-plus boundary-value problems for partial equations; (IV) boundary-value problems for partial equations; and (V) integral and functional equations.

A second and longer look reveals the wealth of material covered. Under I we find first the necessary background material of difference formulas and quadrature formulas; then come simple methods of summation, error estimates, the Runge-Kutta methods, methods using finite differences, Taylor's series, quadrature formulas, the "extrapolation" process, the "interpolation" process, and methods of iteration. Each of these is applied to both first and higher order equations and the error for each is examined. Under II appear linear boundary value problems, use of finite differences, relaxation, iteration, the minimum principle, the Ritz method, least square methods, Galerkin's method, step-by-step methods, and methods for calculating eigenvalues.

Chapter III takes up those partial differential equations, principally of parabolic or hyperbolic type, where the initial values are known at $t=0$, where the space boundary values are given, and for which the solution is computed step-by-step for successive values of t. The method of difference equations is applied to the values of the unknown at lattice points of a rectangular net and the dangers due to incorrect formulation of the difference equation are vividly exhibited. Further topics include choice of the mesh size, propagation of errors, graphical constructions for solving the difference equations, and the use of characteristics.

In IV the author takes up those partial differential equations, mainly of elliptic type, where the solution is not obtainable by a step-by-step process with respect to one variable, but is found as the solution of a set of simultaneous equations. The method of difference equations for a square lattice is explained and applied to such examples as the Dirichlet problem. Further topics include proof of existence and uniqueness of a solution, estimate of error, convergence of the iteration process for solving the equations, relaxation, triangular and hexagonal nets, method of least squares, the Ritz method, Trefftz' method, use of series of known functions, eigenfunctions, and eigenvalues.

Chapter V deals mainly with integral equations, and explains methods where the integrals are replaced by sums, methods of iteration, use of series, cases of singular integral equations, Volterra's equation, and closes with some examples of functional equations.

The text is enlivened by interesting numerical examples, some short biographical notes, and appropriately has as its frontispiece a photograph of C. Runge. *W. E. Milne.*

***Curtiss, John H. Sampling methods applied to differential and difference equations.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 87–109. International Business Machines Corp., New York, N. Y., 1950.

The author presents a part of the theory dealing with the use of probability methods for numerical solution of certain differential equations. A brief historical survey of the study of problems arising in probability theory through corresponding differential equations is followed by a discussion of the "Monte Carlo" method where the point of view is reversed: the attempt being to obtain solutions of given differential or integro-differential equations by suitable statistical experiments, or to study certain physical problems directly by probabilistic models without recourse to the equations describing them.

The paper summarizes some of the recent work in this direction. The diffusion equation is discussed; attention centers mainly on Dirichlet's problem. Random walks leading to the differential equation are generalized in various ways and their properties are studied. The degree of approximation of solutions of the difference equations to those of the differential equation is estimated. The dispersion of the statistical estimates together with the economy of various sampling procedures is discussed. It is followed by description and comparison of bookkeeping procedures necessary for the evaluation of the statistical experiments performed on computing machines. *S. Ulam.*

Bieberbach, Ludwig. On the remainder of the Runge-Kutta formula in the theory of ordinary differential equations. Z. Angew. Math. Physik **2**, 233–248 (1951).

This paper presents a serviceable bound for the error committed when the system $dy_i/dx = f_i(x, y_1, \cdots, y_n)$, $i = 1, 2, \cdots, n$, is solved by the well-known Runge-Kutta method. Special estimates are given for the differential equation of nth order, with particular attention to the cases $n = 2$ and $n = 3$.

W. E. Milne (Corvallis, Ore.).

Lotkin, Max. On the accuracy of Runge-Kutta's method. Math. Tables and Other Aids to Computation **5**, 128–133 (1951).

The author obtains a bound for the error in Kutta's fourth order method (generally known as the Runge-Kutta method) which he considers to be somewhat better than that cited by Bieberbach.

W. E. Milne (Corvallis, Ore.).

Mohr, Ernst. Über das Verfahren von Adams zur Integration gewöhnlicher Differentialgleichungen. Math. Nachr. **5**, 209–218 (1951).

The present paper is designed to correct and supplement an earlier paper on the same subject by W. Tollmien [Z. Angew. Math. Mech. **18**, 83–90 (1938)]. Its aim is to establish a bound for the error committed in Adams' method applied to the numerical solution of the differential equation $y' = f(x, y)$. The function $f(x, y)$ is assumed to have continuous derivatives of fourth order in the region under consideration, and the difference table for y' is truncated to differences of the third order. With these hypotheses a bound for the error is given, which varies as the fourth power of the length of the step-interval. *W. E. Milne.*

Weissinger, Johannes. Eine verschärfte Fehlerabschätzung zum Extrapolationsverfahren von Adams. Z. Angew. Math. Mech. **30**, 356–363 (1950). (German. English, French, and Russian summaries)

The paper deals with various estimates for the error committed in the numerical solution of differential equations by Adams' method. The author presents in detail the estimate of v. Mises, that of Tollmien and Fricke, and then proposes an improvement of v. Mises' result.

W. E. Milne (Corvallis, Ore.).

Vlasov, I. O., and Čarnyĭ, I. A. On a method of numerical integration of ordinary differential equations. Akad. Nauk SSSR. Inženernyĭ Sbornik **8**, 181–186 (1950). (Russian)

The authors present and illustrate with several numerical examples a simple method of solving ordinary differential equations numerically. For a single equation of first order $dy/dx = f(x, y)$ the step from x_0 to x_1 is computed by the formula

$$y_1 = y_0 + \frac{[f(x_0, y_0) + f(x_1, y_0)](x_1 - x_0)}{2 - f_y(x_1, y_0)(x_1 - x_0)}.$$

For systems of equations the step is made with obvious generalizations of this formula. *W. E. Milne.*

Ludwig, Rudolf. Analytische Untersuchungen und konstruktive Erweiterungen zu den graphischen Integrationsverfahren von Meissner und Grammel. Z. Angew. Math. Mech. **31**, 120–130 (1951). (German, English, French, and Russian summaries)

The author investigates analytically two graphical methods for solving differential equations, one due to Grammel and the other to Meissner. The error in each case is examined, with results which show in Grammel's case how to keep the error as small as possible, and in Meissner's case lead to new and improved constructions. For these the error is of the same order as for Simpson's rule.

W. E. Milne (Corvallis, Ore.).

Ludwig, Rudolf. Graphische Integrationsverfahren für die Meissnerschen Linienbilder bei Differentialgleichungen 1. Ordnung. Z. Angew. Math. Mech. **31**, 169–178 (1951). (German. English, French, and Russian summaries)

In an earlier paper [see the preceding review] the author gave methods for determining good approximations with Meisner's graphical method. This however does not apply to a single equation of first order. The present paper exhibits graphical methods suitable for an equation of first order and provides an estimate of error. *W. E. Milne.*

Ludwig, Rudolf. Graphische Integrationsverfahren für die Meissnerschen Linienbilder bei Differentialgleichungssystemen und Differentialgleichungen n-ter Ordnung. Z. Angew. Math. Mech. **31**, 201–208 (1951). (German. English, French, and Russian summaries)

The graphical methods treated in earlier papers [see the two preceding reviews] are here extended to the case of systems of n first order differential equations and to equations of nth order. Error estimates are furnished and specific examples are worked out graphically. *W. E. Milne.*

Fehlberg, Erwin. Bemerkungen zur Entwicklung gegebener Funktionen nach Legendreschen Polynomen mit Anwendung auf die numerische Integration gewöhnlicher linearer Differentialgleichungen. Z. Angew. Math. Mech. 31, 104–114 (1951). (German, English, French, and Russian summaries)

The author determines the coefficients in the expansion of $f(t)$ in a series of Legendre polynomials by the following device. He divides the interval $(-1, 1)$ into ten equal subintervals of length 0.2. In each subinterval he represents $f(t)$ by Newton's interpolation formula of fourth degree formed with the five points obtained by dividing the given subinterval into four equal parts. The coefficient a_n of $P_n(t)$ in the Legendre expansion is formed by using the above approximation for $f(t)$ in the formula

$$a_n=\tfrac{1}{2}(n+1)\int_{-1}^{1} f(t)P_n(t)dt.$$

This leads to a group of integrals, fifty in all, having as integrands the Legendre polynomial $P_n(x)$ times the factorial polynomial occurring in Newton's formula, with limits determined by the particular subinterval. All these integrals have been evaluated and are tabulated in the paper for $n=0, 1, \cdots, 12$. Symmetry reduces the required number of entries to 300. In the second part of the paper the foregoing expansion is applied to the solution of a second order linear differential equation with the boundary values given at two end points of the interval. The coefficients are first expanded in Legendre's series as shown above, and the desired solution is set up as a Legendre's series with unknown coefficients. The problem thus reduces to the solution of a system of $n+1$ simultaneous algebraic equations. The author apparently believes that this round-about method of solution is superior to direct step-by-step numerical solution of the linear differential equation. *W. E. Milne.*

Brock, J. E. An iterative numerical method for nonlinear vibrations. J. Appl. Mech. 18, 1–11 (1951).

The author explains and illustrates by examples a method of iteration for finding the particular solution of $\ddot{x}+g(x,t)=0$ that satisfies $x(0)=A$, $\dot{x}(0)=0$. Actually the desired quantity is τ, the first zero of $x(t)$, which is assumed to exist. Accordingly, the method of iteration is specialized so as to lead quickly to a value of τ. *W. E. Milne.*

Sponder, Erich. Ein zeichnerisches Lösungsverfahren für Differentialgleichungen zweiter Ordnung. Elemente der Math. 6, 53–58 (1951).

The author employs certain simple properties of tangents to the quadratic parabola to provide a means for approximate construction of successive tangents to a given integral curve of $y''=f(x, y, y')$. The method is illustrated by examples. *W. E. Milne* (Corvallis, Ore.).

Landauer, Rolf. Reflections in one-dimensional wave mechanics. Physical Rev. (2) 82, 80–83 (1951).

The author aims at supplying a more accurate solution of the wave equation based on the WKB method. His solution may be given a simple physical interpretation. The potential function is approximated by a staircase function and the wave equation is solved for each interval in which the potential is constant, whereupon the interval-solutions are matched to each other. The waves generated at the discontinuities are then taken into account and are used to obtain second, etc. approximations. The limitations of the WKB method are discussed. Finally, the eigenvalues connected with different periodic potentials are considered.

M. J. O. Strutt (Zurich).

Leutert, Werner. On the convergence of approximate solutions of the heat equation to the exact solution. Proc. Amer. Math. Soc. 2, 433–439 (1951).

In order to illustrate his finite difference methods, L. F. Richardson [Philos. Trans. Roy. Soc. London. Ser. A. 210, 307–357 (1911)] replaced the differential equation in the initial value problem

$$u_t=u_{xx} \text{ for } t>0, \quad 0<x<1,$$
$$u(0,t)=u(1,t)=0, \quad u(x,0)=f(x)$$

by the difference equation

$$v(x,t+\Delta t)-v(x,t-\Delta t)$$
$$=[v(x+\Delta x,t)+v(x-\Delta x,t)-2v(x,t)]2\Delta t/(\Delta x)^2$$

and tried to solve the latter by a step-by-step method. G. G. O'Brien, M. A. Hyman and S. Kaplan [J. Math. Physics 29, 223–251 (1950); these Rev. 12, 751] pointed out that this method is practically worthless, since it is unstable, no matter what value is chosen for the mesh-ratio $r=\Delta t/(\Delta x)^2$. By this it is meant that any numerical error will have a tendency to increase as the step-by-step computation progresses. In Richardson's method the values of the solution on the line $t=\Delta t$ have to be chosen by some different scheme, which is bound to introduce an additional error right at the beginning of the calculations. In the present paper it is shown that, in spite of these facts, the exact solution of Richardson's difference equation will converge, for all constant values of r, to the solution of the original initial value problem, as $\Delta x\to 0$, provided the initial values of the solution of the difference equation are chosen so as to tend in a certain specified way to those of the differential problem. The proof uses the explicit Fourier series expressions for the solutions of the two problems. *W. Wasow.*

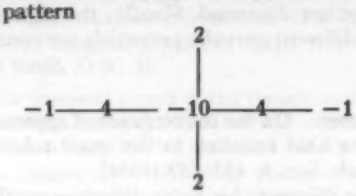
Kuznecov, P. I., and Claf, L. Ya. On the numerical solution of some problems of mathematical physics. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 2, 182–190 (1948). (Russian)

The authors apply the familiar methods of finite differences to find the solution at the nodes of a rectangular net for second order partial differential equations, with dependent variable z and independent variables x, y. The following cases are treated: Case I for which the boundary conditions are $z=\varphi_0(x)$, $\partial z/\partial y=\varphi_1(x)$ on the x-axis. Case II for which the boundary conditions provide the values of z and its normal derivative on a plane curve $y=f(x)$. Case III is the plane Dirichlet problem. As an example the plane wave equation is solved with given initial conditions. [A much shorter method could have been used.] *W. E. Milne.*

Allen, D. N. de G., and Severn, R. T. The application of relaxation methods to the solution of non-elliptic partial differential equations. I. The heat-conduction equation. Quart. J. Mech. Appl. Math. 4, 209–222 (1951).

In order to apply the method of relaxation to the heat equation $\partial v/\partial t=\kappa\partial^2 v/\partial x^2$ the authors make the transformation $v=\partial w/\partial t+\kappa\partial^2 w/\partial x^2$ which leads to the equation $\partial^2 w/\partial t^2-\kappa^2\partial^4 w/\partial x^4=0$.

The finite-difference approximation to this equation for a rectangular net, with proper choice of Δx and Δt, gives rise to the pattern



a form to which relaxation is applicable. The boundary conditions must also be transformed, and extra ones added.

Suppose that $v=0$ for $t=0$, but for $t>0$, $v=0$ at $x=0$, $v=100$ at $x=L$. The new conditions may be given as

$$\begin{aligned} w&=0, \quad \partial^2 w/\partial x^2=0, && \text{at} \quad x=0,\\ w&=0, \quad \kappa\partial^2 w/\partial x^2=100, && \text{at} \quad x=L,\\ \partial w/\partial t+\kappa\partial^2 w/\partial x^2&=0, && \text{when} \quad t=0,\\ \partial w/\partial t&=0, && \text{when} \quad t=T, \end{aligned}$$

where T is the time for which the solution is required.

W. E. Milne (Corvallis, Ore.).

Slobodyanskiĭ, M. G. The determination of the derivatives of the unknown functions in the solution of problems by the method of finite differences. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 245–250 (1951). (Russian)

In certain problems requiring the solution w of a partial differential equation the desired quantity is not w itself but one or more partial derivatives of w. When an approximate solution is obtained by the use of a lattice and a difference equation, the derivatives can be found by difference formulas, but the degree of accuracy is less than for w itself. The author shows how to set up partial differential equations satisfied by the desired derivatives and which can be solved, for a given lattice, to the same degree of accuracy as the equation for w. The idea is applied to the torsion of a rod of square cross section, and to the case of a square loaded plate. *W. E. Milne* (Corvallis, Ore.).

Spencer, Domina Eberle, and Stakutis, Vincent J. The integral-equation solution of the daylighting problem. J. Franklin Inst. **252**, 225–237 (1951).

Birkhoff, G., Young, D. M., and Zarantonello, E. H. Effective conformal transformation of smooth, simply connected domains. Proc. Nat. Acad. Sci. U. S. A. **37**, 411–414 (1951).

This paper is a summary of a longer article which is to appear later. Two numerical methods for approximating the function which maps a convex region conformally on the interior of a circle are outlined. The first involves the reduction of the mapping problem to a non-singular linear integral equation similar to an equation given by Carrier. The second is a modification of the Theodorsen-Garrick method which uses the curvature and the tangent direction of the boundary of the region to be mapped instead of the polar coordinates of the boundary. A method analogous to this one is also stated for calculating steady, non-viscous incompressible flows with free boundaries. *C. Saltzer.*

Yuškov, P. P. The practical harmonic analysis of empirical functions when the given curve is replaced by another approximating the given one in contour. Akad. Nauk SSSR. Inženernyĭ Sbornik **6**, 197–210 (1950). (Russian)

The author proposes various procedures for the approximate determination of the coefficients in the Fourier series for a given function. A typical procedure is to divide the interval $(0, 2\pi)$ into n equal parts, replace $f(x)$ by a sequence of parabolic arcs joined continuously at the division points x_k, and compute the Fourier coefficients for the new function. To carry this out numerically one first computes the coefficients usually found in practical harmonic analysis, namely

$$\alpha_m=\frac{2}{n}\sum_{k=0}^{n-1} f(x_k)\cos\frac{2m\pi k}{n},$$

$$\beta_m=\frac{2}{n}\sum_{k=0}^{n-1} f(x_k)\sin\frac{2m\pi k}{n},$$

and then gets the desired coefficients a_m and b_m by using correction factors c_m, such that $a_m=c_m\alpha_m$, $b_m=c_m\beta_m$. These correction factors can be computed once for all and sets of them for several different cases are supplied by the author.

W. E. Milne (Corvallis, Ore.).

Banachiewicz, T. On the general least squares interpolation formula. Acta Astr. Sér. C. **4**, 123–128 (1950).

The author explains and illustrates by numerical examples the use of cracovians for setting up a set of orthogonal functions to obtain the best expression in the sense of Least Squares for fitting a function y given by its values $y_1, y_2, \cdots, y_n$ at n values of the argument $x_1, x_2, \cdots, x_n$.

W. E. Milne (Corvallis, Ore.).

Wolf, H. Ist eine Revision der Gewichtskoeffizientenberechnung in der Methode der kleinsten Quadrate erforderlich? Z. Angew. Math. Mech. **31**, 153–155 (1951).

This paper examines the objection raised by Piotrowski [Proc. Nat. Acad. Sci. U. S. A. **34**, 23–26 (1948); these Rev. **9**, 381] to the usual method of calculating weights for nonlinear problems and concludes that in many practical cases the objection is without foundation. *W. E. Milne.*

Corpaciu, Alexandre J. Considérations sur les équations de condition et le calcul des développements des corrélatives dans la compensation conditionnelle des chaînes de quadrilatères avec deux diagonales. Bull. Géodésique **1951**, 163–169 (1951).

Kneissl, M., et Strasser, G. Zur Auswahl der Dreiecksketten und Bestimmung der Längen- und Richtungsübertragungsfehler. Bull. Géodésique **1951**, 123–162 (1951).

Athen, Hermann. Automatische Grossrechenmaschinen. Elektrotech. Z. **72**, 518–522 (1951).

Bückner, Hans. Über die grossen Rechengeräte. Abh. Math. Sem. Univ. Hamburg **17**, 22–68 (1951).

This paper is an exposition of the current state of the computational equipment field, largely from the logical point of view. The author discusses not only digital machines, but also analog ones. There are two main parts to the paper, one devoted to each of these types of instruments in addition to a brief historical introduction. In the part devoted to analog machines he describes the integrator, the function table, the adder, and the methods for interrelating these components into a differential analyser to integrate differential equations. This part also describes the state of the technical development in Germany of analysers. In the part devoted to digital machines the author discusses a

hypothetical machine with a given repertoire of orders and shows in some detail how codes can be built out of these orders. He closes with a few remarks on some machines being built in America and with some observations on engineering details. *H. H. Goldstine* (Princeton, N. J.).

Bückner, Hans. Bericht über die modernen Rechengeräte. Jber. Deutsch. Math. Verein. Abt. 1. **55**, 15–38 (1951).

Bückner, Hans. Zum Zirkeltest der Integrieranlagen. Z. Angew. Math. Mech. **31**, 224–226 (1951).

The author shows how the backlash in a differential analyzer can be determined by setting up the system $\dot{z}_1=-z_2$, $\dot{z}_2=z_1$. Because of backlash the output of the differential analyzer is a spiral whose rate of growth per turn is proportional to the radius and the total backlash. The backlash is in the form $a \operatorname{sgn} \dot{z}$ except near a reversal of direction. This is entered into the equation as a forcing term. The author also shows that another type of mechanical error does not affect the closure of the answer curve but does not consider slip. *F. J. Murray* (New York, N. Y.).

Stojaković, Mirko. Über einige neue mechanische Analogien der arithmetischer Grundoperationen. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 51–60 (1950). (Serbo-Croatian. German summary)

Schminke, H. Eine einfache Vorrichtung zur Addition von Produkten. Z. Angew. Math. Mech. **31**, 297 (1951).

Artobolevskiĭ, I. I. A mechanism for raising to a cube. Doklady Akad. Nauk SSSR (N.S.) **79**, 933–935 (1951). (Russian)

Barnes, R. C. M., Cooke-Yarborough, E. H., and Thomas, D. G. A. An electronic digital computor using cold cathode counting tubes for storage. II. Electronic Engrg. **23**, 341–343 (1951).

Robinson, J.-R., et Courbon, J. Le calcul numérique à la machine en résistance des matériaux. Ann. Ponts Chaussées **121**, 421–462 (1951).

RELATIVITY

Blanuša, Danilo. Die Grundlagen der relativistischen Kinematik. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 1–32 (1951). (Serbo-Croatian. German summary)

Starting with a set of postulates which are valid also in classical mechanics (the existence of inertial systems, the isotropy of space, Newton's first law of motion, the homogeneity of time, and the relativity principle), the author derives the transformation equations connecting the Cartesian coordinates and the time in one inertial system with those in another inertial system. These equations are linear and have the same form as those of the Lorentz transformation except for the fact that $1/c^2$ (where c is the speed of light in free space) is replaced by an undetermined parameter k. For $k=0$ one gets the Galilean transformation of classical mechanics. The author obtains expressions for the resultant of two velocities, the length of a moving rod and the proper time of a moving clock in terms of the parameter k. In principle the latter could be determined by means of a kinematics experiment, e.g., by a measurement of the length of a moving rod, although this would not be practicable at the present time. Setting $k=1/c^2$ leads to the formalism of the special relativity theory. *N. Rosen.*

***Landau, L., and Lifshitz, E. The Classical Theory of Fields.** Translated from the Russian by Morton Hamermesh. Addison-Wesley Press, Inc., Cambridge, Mass., 1951. ix+354 pp. $7.00.

This is a translation of the authors' Teoriya polya [OGIZ, Moscow-Leningrad, 1948] which was reviewed in these Rev. **11**, 410.

Tonnelat, Marie-Antoinette. Compatibilité des équations de la théorie unitaire des champs. C. R. Acad. Sci. Paris **232**, 2407–2409 (1951).

In Einstein's "hermitian" unified field theory and in similar theories the equation which determines the affine connection $\Delta^{\sigma}_{\mu\nu}$ in terms of the non-symmetric field tensor density $R^{\mu\nu}$ is

$$R^{\overline{\mu\nu}}{}_{:\rho}=0, \tag{1}$$

where $:\rho$ denotes a covariant derivative with respect to the affine connection. The author shows that these equations determine $\Delta^{\sigma}_{\mu\rho}$ and hence obtains the only value of this quantity compatible with (1). The author then shows that if the affine connection in (1) is replaced by another one then one may obtain solutions of (1) compatible with $\Delta^{\sigma}_{\mu\rho}=0$. *A. H. Taub* (Urbana, Ill.).

Tonnelat, Marie-Antoinette. Les équations du champ unitaire et leurs approximations. C. R. Acad. Sci. Paris **233**, 513–516 (1951).

The author evaluates the equations $R_{\mu\nu}=\lambda r_{\mu\nu}$ for various choices of $R_{\mu\nu}$ as a function of an affine connection, where the affine connection is determined by the condition $r_{\mu\nu;\rho}=0$ and it is assumed that the antisymmetric part of $r_{\mu\nu}$ is such that terms containing products of three or more components may be neglected in comparison to unity. *A. H. Taub.*

Tonnelat, Marie-Antoinette. Les équations électromagnétiques déduites d'une théorie unitaire des champs. C. R. Acad. Sci. Paris **233**, 555–557 (1951).

The author points out some differences between previously obtained approximate equations [see the paper reviewed above] for the antisymmetric part of the field tensor and Maxwell's equations. It is assumed that the symmetric tensor representing the gravitational field is a metric tensor in a space of constant curvature. *A. H. Taub.*

Jonsson, C. V. Studies on five-dimensional relativity theory. Ark. Fys. **3**, 87–129 (1951).

In the first part of this paper the author evaluates the field equations of Klein's five-dimensional unified field theory in a coordinate system in which the components γ_{00}, γ_{m0} of the fundamental tensor $\gamma_{\mu\nu}$ are independent of the fifth coordinate x^0. In the second part he linearizes these equations and shows that they describe fields of spin $2\hbar$. These equations are then quantized and it is shown that when matter is introduced and described by spinor particles of half-integer spin then the interaction terms obtained are of the form occurring in meson pair theories.

A. H. Taub (Urbana, Ill.).

Kohler, Max. Zum Problem der Planetenbewegung nach der allgemeinen Relativitätstheorie. Z. Physik **130**, 139–143 (1951).

The author shows that neither Schwarzschild's spherically symmetric metric in general relativity,

$$ds^2=\frac{dr^2}{1-\alpha/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)-(1-\alpha/r)dx_4^2,$$

where $x=ct$ and $\alpha=2GM/c^2$, nor its first approximation,

$$ds^2=\left(1+\frac{\alpha}{r}\right)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)-\left(1-\frac{\alpha}{r}\right)dx_4^2,$$

can be obtained merely by a transformation of coordinates from a pseudo-Euclidean line element. An explicit appeal to Einstein's field equations or to an equivalent variational principle is needed. The Schwarzschild solution cannot therefore be derived solely from the principle of equivalence, and consequently the author claims that the relativistic motion of perihelion obtained by Einstein depends essentially on his field equations. *G. J. Whitrow.*

Fihtengol'c, I. G. On the antisymmetric moment of momentum tensor. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 648–651 (1951). (Russian)

In section 13 of their book, The Theory of Fields [2d ed., Moscow-Leningrad, 1948; these Rev. **11**, 410], Landau and Lifšic assert that $N^{\mu\nu}=\sum_i(x_i^\mu P_i^\nu-x_i^\nu P_i^\mu)$ is a constant antisymmetric tensor if P_i^μ represents the linear momentum, and x_i^μ the position, of the ith particle and the summation is over all the particles of an isolated system. Fihtengol'c points out that their definition is only valid for non-interacting particles. Using the approximate expression for P_i^μ given in his previous paper [same Žurnal **20**, 233–242 (1950); these Rev. **11**, 746] for interacting particles he finds that $N^{\mu\nu}$ is constant, and to the approximation involved, behaves as a tensor under transformations from one harmonic coordinate system to another. *A. J. Coleman.*

Galli, Mario. Vitesse de phase et vitesse de groupe dans l'optique relativiste. Rev. Optique **30**, 174–184 (1951).

The author disagrees with the conclusion of R. W. Ditchburn [Rev. Optique **27**, 4–14 (1948); these Rev. **9**, 539] that the relation $c_pc_g=c_1^2$ is required by the special theory of relativity but that the relations $c_p=c_g=c_1$ are not; here c_p and c_g are respectively the phase- and group-velocities of light in vacuo and c_1 the constant occurring in the fundamental form $dx^2+dy^2+dz^2-c_1^2dt^2$. To the reviewer the disagreement appears to be essentially a disagreement as to the meaning of the words "theory of relativity". The author quotes its principles as involving (i) the invariance of natural laws for all inertial reference systems, and (ii) the absolute constancy of the velocity of light for an observer belonging to such systems; he then remarks that this second principle is hardly comprehensible if one admits that the phase- and group-velocities are different and functions of the frequency. If one starts from this basis, Ditchburn's conclusion is not debatable, because it is meaningless. However, an interpretation of the "theory of relativity" as a theory based solely on Lorentz invariance (and not tied up from the beginning with light propagation) is now widely accepted, and with such an interpretation the author's criticism seems no longer valid. The paper includes some remarks on the Schrödinger equation and on De Broglie's wave theory of the photon possessing rest-mass.

J. L. Synge (Dublin).

Bleksley, A. E. H. A new approach to cosmology. III. South African J. Sci. **48**, 20–24 (1951).

Parts I and II appeared in the same J. **46**, 343–346 (1950); **47**, 13–15 (1950); these Rev. **12**, 449.

MECHANICS

***Banach, Stefan. Mechanics.** Translated by E. J. Scott. Monografie Matematyczne. Tom XXIV. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951. iv+546 pp.

This book contains the author's lectures in mechanics, given for many years at the Jan Kazimierz University and the Lwów Polytechnic Institute. It is a textbook on the mechanics of particles and rigid bodies, well printed and illustrated with diagrams, covering the traditional field for general university requirements, including those of engineering students. The chapters deal with theory of vectors, kinematics of a point, dynamics of a material point, geometry of masses, systems of material points, statics, kinematics and dynamics of a rigid body, virtual work, dynamics of holonomic systems, variational principles of mechanics. The English-speaking reader must of course accommodate himself to the use of left-handed axes, and this is no great hardship; however this is combined with the rather confusing convention that a right-handed rotation is positive, so that the formula for the velocity of a point of a spinning body is $\mathbf{v}=\mathbf{r}\times\boldsymbol{\omega}$ instead of $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$, as given in English texts and also in French ones, where the left-handed axes are used with the more natural convention that a left-handed rotation is positive. The confusion in the reader's mind is increased by a misprint in line 9, p. 46, where the order in the vector product is reversed. Engineering students develop their understanding of mechanics by working through many special problems; the large amount of the text devoted to the solution of such problems should give them the necessary experience, although it might have been well to include some unsolved exercises to test their power. But anyone interested in the logical structure of mechanics turns to the general theory underlying the problems, and, although the arguments are in general clearly and explicitly stated, one can hardly accept the statement (p. 73) that if a material point is at rest, then it is said to be in equilibrium. The same statement is repeated (p. 234) with reference to a body. On p. 287 it is stated that if a system of rigid bodies is in equilibrium, the sum and total moment of the external forces are zero; this is followed by the statement that this condition is only sufficient, but not necessary for equilibrium (sufficient and necessary should be interchanged). The potential V is defined so that force is its gradient. Thus, when potential energy is introduced, it is $-V$, and the kinetic potential is $W=E+V$ and the Hamiltonian $H=E-V$, E being the kinetic energy. This deviation from what is now a fairly well standardised notation seems unnecessary. But apart from these criticisms, the book contains a complete and careful exposition of mechanics within the bounds set. In the chapter on variational principles, the treatment of variations is more logical than is usual in comparable books. One notational device seems worthy of general adoption: differentiation with respect to the time is indicated by a dot in a superscript position following the differentiated quantity, instead of above it.

J. L. Synge (Dublin).

*Goldstein, Herbert. **Classical Mechanics.** Addison-Wesley Press, Inc., Cambridge, Mass., 1951. xii+399 pp.

This book is an attempt at an exposition of classical mechanics which can serve as a preparation for study in modern physics. Therefore much attention is paid to those topics which are related to quantum mechanics and to those formulations which are of importance for modern physics. The starting point for the dynamics of systems is d'Alembert's principle which leads to Lagrange's equations (chapter I), Hamilton's principle (chapter II), Hamilton's equations and the principle of least action (chapter VII). Velocity-dependent potentials and forces, such as electromagnetic forces on moving charges have been included as well as an extension of Hamilton's principle to certain non-holonomic systems. Chapter III, devoted to central force motion, includes a classical treatment of scattering problems. Chapter IV discusses the kinematics of rigid body motion; it contains a treatment of Euler angles and the Cayley-Klein parameters in connection with spinors. The rotations are treated from the standpoint of matrix transformation, and Euler's theorem on the motion of a rigid body is presented in terms of the eigenvalue problem for an orthogonal matrix. The principal moments of inertia and the principal axes are also obtained from an eigenvalue problem. The case of multiple roots, however, is not considered in detail. As examples of rigid body motion are given the force-free motion, the heavy symmetrical top and the motion of a charged spinning body in a uniform magnetic field. In chapter VI a treatment is given of the equations of motion in special relativity. A Lagrangian formulation of certain relativistic problems is presented. Canonical transformations and Lagrange and Poisson brackets are discussed in chapter VIII. Considerable space has been devoted to the Hamilton-Jacobi theory and the action angle variables. The connection between Hamilton-Jacobi theory and geometrical optics is shown (chapter IX). Chapter X is devoted to small oscillations, free vibrations as well as forced vibrations. The proof that two quadratic forms can be diagonalised together is carried out only in the case of unequal roots of the secular equation. For the case of multiple roots the author refers to other books. Chapter XI discusses the transition from discrete to continuous systems. An elementary description is given of the Lagrangian and Hamiltonian formulation of fields. The author thinks it necessary to comment upon the references listed at the end of each chapter.

J. Haantjes (Leiden).

Kolčin, N. I. Applications of the formulas for a worm gear to special cases of meshing. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **5**, no. 19, 82–102 (1948). (Russian)

This is the second part of the author's paper on skew worm gears [same Trudy **3**, no. 9, 18–51 (1947); these Rev. **13**, 172] applying its formulas to the case of an involute helicoid worm meshing with a straight rack. He determines the contact lines (they are straight), the surfaces, and the angles of meshing. The same is done for a screw helicoid worm. The author uses the same equations to determine the surfaces of a hob and a disk-shaped milling cutter for involute and screw helicoids. All equations are parametric in cartesian coordinates. There are eight pages of numerical examples and graphs.

A. W. Wundheiler (Chicago, Ill.).

Riftin, L. P. Analytic design of computing cams with two degrees of freedom. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **8**, no. 32, 5–30 (1950). (Russian)

One half of this paper deals with elementary considerations of two-dimensional cam design. The other half is devoted to the pressure angle defined as the angle between the absolute and relative velocities of the tip of the follower. This pressure angle is expressed in terms of the position and velocity parameters for sliding or rotating plane cams with sliding or rocking followers (flat or edged) and for two-dimensional sliding or helical cams with sliding or rocking followers. *A. W. Wundheiler* (Chicago, Ill.).

Kosticyn, V. T. Determination of minimum cam dimensions for straight-line followers. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **3**, no. 12, 23–63 (1947). (Russian)

The connection between the minimum size of a cam and prevention of frictional self-locking is examined. Assuming the contact between the follower and the guides to occur at four vertices of a rectangle of width d and length l, the distance of its center from the center of the roller being y, the formula, $\cot \gamma_{cr} = f(2y+l \mp fd)/l$, is derived for the minimum γ_{cr} of the pressure angle γ (between the cam normal and follower axis) at which self-locking occurs. The sign depends on the direction of motion; f is the coefficient of friction. The critical angle is a minimum for the lowest position of the follower. Let γ_0 be smaller than this min γ_{cr} by a prescribed safety margin. Then $\gamma_{max} \leqq \gamma_0$ is a design condition limiting the size of cam. From the relation $\tan \gamma = v/\rho\omega$ (ρ is the distance between the center of cam and roller) an expression for γ is derived. The cases of constant speed, constant acceleration, and simple harmonic motion are studied. In all these cases γ_{max} proves to be a decreasing function of the minimum cam radius R, and the design condition is found to provide a lower limit for R in terms of several design constants. The author concentrates on the dependence of R on l, and determines min $(R+l)$ as the answer to the over-all dimensions problem. Numerical examples are generous. For the general cam the author offers pure and mixed graphical procedures, extended also to followers with an offset. These procedures claim more than one-half of the paper, and are given detailed presentation. For a continuation of this work by the same author see Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1531–1537; these Rev. **10**, 489. *A. W. Wundheiler.*

Tišin, M. M. The profiling of the cams of the gas distribution of aviation motors. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **1**, 217–239 (1947). (Russian)

Four cam mechanisms with rectilinear or pivoted followers (flat-faced or roller equipped) are discussed. The symmetric cam profile is composed of circular arcs AA', AB, BC, CC', $C'B'$, and $B'A'$. The arcs AA' and CC' are centered at the center O of the cam; AB and $B'A'$ are concave, BC and $C'B'$ are convex relative to O. The initial and final accelerations of the follower are given, as well as the lift of the follower and the radius of the initial arc of the cam. The accelerations are expressed in terms of the position and design parameters of the mechanism. These expressions lead to design conditions in the form of equations which can be solved numerically for the angles of the arcs involved. Graphs and numerical examples are provided. *A. W. Wundheiler.*

Bereis, R. Aufbau einer Theorie der ebenen Bewegung mit Verwendung komplexer Zahlen. Österreich. Ing.-Arch. **5**, 246–266 (1951).

The classical geometry of motion in a plane is developed neatly by means of the complex variable and exhibited by graphical constructions. The equations and constructions of the tangent, the osculating circle, the affine normal (limit of bisector of chord parallel to tangent and passing through point on curve), the tangent parabola and the osculating conic are shown. The loci of the center of rotation, the center of velocity, the center of acceleration and higher poles are derived. The inflection circle, the Ball point, the locus of umbilical points and the Burmester points are obtained. If, at each point in the motion, a vector is drawn for each nth derivative, but rotated $2-n$ right angles, the terminus of the chain of vectors corresponds to the nth pole. This chain of poles is used to approximate desired motions.

M. Goldberg (Washington, D. C.).

Prosciutto, Aristide. Sul problema generale dell'equilibrio di un corpo rigido soggetto a vincoli elastici. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **5** (1947/48), 67–76 (1949).

Analizzando le proprietà generali di un sistema di vincoli elastici applicati ad un corpo rigido, sulla base dei concetti vettoriali di "vite" (o "asse elicoidale") e di "motore" (o "diname"), si dimostra l'esistenza di una sestupla di "viti principali ortogonali", che hanno la proprietà di essere simultaneamente sostegni geometrici di "motori-forze" e dei corrispondenti "motori-spostamenti". Si deducono poi altre proprietà caratteristiche generali inerenti ai sistemi di vincoli elastici. *O. Bottema* (Delft).

Kašanin, R. Les équations générales du mouvement d'un système de points matériels. Srpska Akad. Nauka. Zbornik Radova, Knj. **7**. Matematički Institut, Knj. **1**, 17–57 (1951). (Serbo-Croatian. French summary)

The results of this paper were published earlier in French [Acad. Serbe Sci. Publ. Inst. Math. **2**, 116–130 (1948); these Rev. **10**, 489].

Semenov, M. V. The rôle of Russian and Soviet scientists in the theory of regulation of powered machines. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **6**, no. 23, 5–29 (1949). (Russian)

Automatic regulation originated in 1763 with I. I. Polzunov who invented the float used in flush toilets. Then came Watt and Poncelet. The foundations of modern regulation were laid by I. A. Vyšnegradskiĭ (1876) who created the method of small oscillations. Western scientists tried to disparage his work, charging that he neglected dry friction; the author refutes this criticism. The development was resumed in 1934 (I. N. Voznesenskiĭ and his pupils). In 1938 A. V. Mihaĭlov discovered the stability criterion unfairly credited to A. Leonard. The work of the Moscow seminar on topological mechanics is credited simply to "A. A. Andronov and his pupils"; their method is rated as "the most powerful", and dismissed in twenty-four lines. The author regrets that the students of automatic regulation in aircraft instrumentation did not follow Voznesenskiĭ in reducing their criteria to design-office level. Nevertheless, the Russian results have greatly outdistanced those of all foreign countries. The author makes no mention of the names of Lyapunov, Krylov, and Bogolyubov.

A. W. Wundheiler (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Byušgens, S. S. On stream lines. Doklady Akad. Nauk SSSR (N.S.) **78**, 837–840 (1951). (Russian)

The author continues his earlier work [Izvestiya Akad. Nauk. SSSR. Ser. Mat. **10**, 73–96 (1946); **12**, 481–512 (1948); these Rev. **8**, 90; **10**, 633]. In the notation of this article (with J instead of I), $\delta=\omega_0{}^1\omega_0{}^2\omega_0{}^3\neq 0$, $p=p_\alpha\omega_0{}^\alpha$, $q=q_\alpha\omega_0{}^\alpha$, $r=r_\alpha\omega_0{}^\alpha$, he defines the field of vectors "adjoint" to the field J_3:

$$J=\frac{dJ_3}{ds}-(\operatorname{div} J_3)J_3=q_3J_1-p_3J_2+(p_2-q_1)J_3.$$

When J is normal to a family of surfaces, J_3 is called semi-special; when J is a gradient, J_3 is called special. It is proved (a) that the congruence of stream lines is semi-special, and (b) that $(d\ln V)\omega_0{}^1\omega_0{}^2=(p_2-q_1)\delta$; these two conditions are necessary and sufficient for stream lines. It is also shown that a congruence of parallel helices on parallel cylinders is semi-special and minimal. *D. J. Struik* (Cambridge, Mass.).

Lahaye, Edmond. Contribution à la résolution des équations du mouvement d'un fluide et des équations de la circulation atmosphérique générale. Inst. Roy. Météorolog. Belgique, Mémoires **38**, 79 pp. (1950).

In Part I, Ch. I the author claims to obtain a new method of expressing the velocity $\mathbf{v}$ in a region in terms of the vorticity $\mathbf{w}$, and thence of obtaining a steady general solution of Euler's equations for perfect inviscid fluids in which the velocity vector takes on an assigned steady tangential value at each point of an arbitrary closed surface S. From this he claims to show that it is necessary that $\mathbf{w}=k\mathbf{v}$, $k=$const., and that the strong Bernoulli theorem must hold. Hence he obtains an integral equation for the velocity and claims to prove the existence of a solution. In Ch. II he claims to extend the result to isentropic perfect gas flows. In Ch. IV he claims to obtain a similar integral equation for flow of a viscous incompressible fluid. Part II treats unsteady motions, and Part IV, relative motions.

[Reviewer's note. The results given are important if they are true. The reviewer makes no pretence of having followed in detail the lengthy and elaborate analysis, but he finds even the opening steps quite incomprehensible. The author assumes that S can be written in the form $z=\nu(x, y)$, and in all his "solutions" he uses partial integrals from $\nu(x, y)$ to z, stating that on the surface they vanish. Such integrals are certainly meaningless for any surface S which is not convex, and they are dubious in any case. The author does not state at the outset whether he is considering the exterior or interior problem; for the former, such integrals cannot possibly vanish on all of S. He applies also a mysterious surface condition for which he gives three different forms (eq. (2.8), footnote p. 11, and eq. (3.2)). These do not seem to be equivalent. One of them is $\mathbf{w}\cdot d\mathbf{S}=0$. By Stokes's theorem it follows then that $\mathbf{v}=0$ on S, so that the boundary velocity is by no means arbitrary. In any case it seems most unlikely that the velocity in an inviscid fluid can take on either arbitrary or zero values on a given closed surface. Finally, if we grant all the author's analysis up to p. 11, it does not at all follow that $\mathbf{w}=k\mathbf{v}$ is necessary, but merely that it is sufficient to satisfy some of his conditions. It is a well-known theorem due to Gromeka [Nekotorye slučai dviženiya nesžimaemoĭ židkosti, Kazan, 1881, see Gl. II, §9a] and Beltrami [Opere matematiche, vol. 4, Hoepli, Milan, 1920, pp. 300–309] that if $\mathbf{w}=k\mathbf{v}$, then independently

of whether or not k=const. the strong Bernoulli theorem holds.] *C. Truesdell* (Bloomington, Ind.).

Lahaye, Edm. Une propriété générale du tourbillon dans le mouvement permanent d'un fluide parfait. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 911–931 (1950).

The author claims to obtain again the results of Part I, Ch. II in the paper reviewed above. Now he distinguishes two cases: S is not a cylinder, or it is. For the latter case he claims that in general the motion must be irrotational. The analysis is based on formulae taken from the paper reviewed above, and the results are open to the same objections.

C. Truesdell (Bloomington, Ind.).

van den Dungen, F. H., et Lahaye, Edm. Sur le mouvement permanent relatif d'un fluide parfait. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 992–998 (1950).

The authors claim to show that in case a fluid is in contact with a rigid surface S rotating at angular velocity ω the relation $w+2\omega r=K\rho v$, K=const., holds. The analysis is based on results from the two papers reviewed above.

C. Truesdell (Bloomington, Ind.).

Manwell, A. R. A method of variation for flow problems. II. Quart. Appl. Math. **9**, 405–412 (1952).

This is a restatement, in more general form, of the author's method [Quart. J. Math., Oxford Ser. (1) **20**, 166–189 (1949); these Rev. **11**, 223]. He proves first the main lemma which is essentially that, if J is a functional of the geometry of a closed curve C and of the velocities in the field and on C, then, for variations giving infinitesimal velocity changes everywhere, $\delta J=0$ for maximum or minimum J. The first application is to plane incompressible flow, taking for J the rather general form $A^{-1}\int_C F(v)ds$, where A is the area enclosed by the profile C, v is the speed and the integral is carried around C. The method leads to an integro-differential equation whose solution would determine the mapping of the profile onto a circle. Solutions for some special cases are indicated. Next it is shown how a similar problem with an auxiliary restriction can be handled. Finally, the method is shown to be applicable to a more general class of problems in potential theory, and an electrical problem is solved as an example.

W. R. Sears (Ithaca, N. Y.).

Kaufmann, W. Über den Mechanismus der Wirbelkerne einer Kármánschen Wirbelstrasse. Ing.-Arch. **19**, 192–199 (1951).

The author proposes to explain the observed dissipation of the Kármán vortex street shortly downstream from the body generating the vortices by taking into account the finite dimensions of the vortex cross-section or "kernel". With the assistance of empirical data on the Kármán street and simplifying assumptions concerning the distribution of vorticity in the vortex kernel, he deduces by the energy balance method the dimensions of the kernel. This occupies the major portion of the paper, which concludes with a brief qualitative argument that the finite size of the vortex is not of itself reason for instability of the Kármán street. An exact theory of the stability of the Kármán street for vortices of finite section has been given by Rosenhead [Proc. Roy. Soc. London. Ser. A. **127**, 590–612 (1930)], who concludes stability for strictly two-dimensional disturbances.

D. Gilbarg (Bloomington, Ind.).

Landweber, L. The axially symmetric potential flow about elongated bodies of revolution. David W. Taylor Model Basin, Rep. no. **761**, 61 pp. (1951).

L'écoulement à potentiel et de révolution autour d'un corps de révolution allongé est étudié par deux méthodes: (1) À la suite de Weinig [Z. Techn. Phys. **9**, 39–43 (1928)] l'auteur cherche à realiser l'écoulement au moyen d'une distribution de doublets de densité $m(t)$ sur l'axe de révolution. Cela conduit à l'équation intégrale: $\int_a^b r^{-3}m(t)dt=\frac{1}{2}$. Il faut d'abord déterminer a et b. On les obtient par des développements en série au voisinage de l'avant et de l'arrière du solide. (On est conduit à un système d'une infinité d'équations du premier degré, mais on garde seulement les trois premières inconnues.) En s'inspirant de Weinblum [Ing.-Arch. **7**, 104–117 (1936)] on choisit un densité approchée $m_1(t)$. Cette distribution sert de valeur initiale pour une résolution de l'équation intégrale par itération. La convergence n'est pas etudiée, mais on peut évaluer l'erreur. Un exemple est traité et discuté. (2) u désignant la vitesse en un point de la surface du solide, on peut se ramener à l'équation intégrale $\int_0^P \frac{1}{2}r^{-3}Uy^2ds=1$, P le demi-périmètre du solide. Une répartition initiale de U est choisie et l'équation est ensuite résolue par itération. L'exemple précédent est repris.

J. Kuntzmann (Grenoble).

Nickel, Karl. Über druckpunktfeste Profile. Z. Angew. Math. Mech. **31**, 297–298 (1951).

It is shown, on the basis of thin-airfoil theory for incompressible flow, that every profile having fixed center-of-pressure must possess an inflection point in its mean-camber curve, excepting the trivial case of the uncambered profile.

W. R. Sears (Ithaca, N. Y.).

Belyakova, V. K. Oscillations of a flat plate under a free surface taking account of small quantities of the second order. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 504–510 (1951). (Russian)

N. E. Kočin [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk. **1939**, 37–62 (1939)] in a paper on the two-dimensional problem of oscillations under a free surface obtained on the basis of the linearized theory the anomalous result that a horizontal flat plate oscillating in a vertical direction will, for certain critical values of the frequency, experience a vertical force but generate no waves. By retaining terms through the second order, the author of the paper under review shows that in addition to the waves predicted by the linearized theory there are also other waves of half their period and one fourth their wave length which do not vanish at these critical frequencies.

J. V. Wehausen.

Truesdell, C. Proof that Ertel's vorticity theorem holds in average for any medium suffering no tangential acceleration on the boundary. Geofis. Pura Appl. **19**, nos. 3–4, 3 pp. (1951).

In its differential form, Ertel's vorticity theorem does not hold for viscous fluids. However, an integral form is found to be true in the mean for any volume on whose bounding surface the fluid acceleration is wholly normal, the fluid being arbitrary. As a special case, it is shown that in a domain on whose boundaries the material suffers no tangential acceleration, there is on the average no diffusion of vorticity.

G. C. McVittie (London).

Hatanaka, Hiroshi. On the general solutions of the two-dimensional Oseen's equations of a viscous fluid. Rep. Inst. Sci. Tech. Univ. Tokyo 5, 45–48 (1951). (Japanese. English summary)

In this paper a study is made of the general solutions of the two-dimensional Oseen equations for a viscous fluid in relation to the potential flow of an ideal fluid. Asymptotic behaviors at infinity are also investigated.

Author's summary.

Baranov, G. G. On the solution of Reynolds' equation for a bearing of finite length. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 1, 153–178 (1947). (Russian)

L'auteur utilise la méthode de Reynolds, moins exacte que celle de Joukovsky, mais applicable aux coussinets de longueur finie. Soit

$$(1)\qquad \frac{\partial}{\partial\alpha}\left[(\epsilon-\cos\alpha)^3\frac{\partial p}{\partial\alpha}\right]+r^2(\epsilon-\cos\alpha)^3\frac{\partial^2 p}{\partial z^2}+h\sin\alpha=0$$

l'équation de Reynolds, avec p = pression hydrodynamique, α = angle de position, z, $-b<z<b$, la distance d'une section normale du centre de l'axe du coussinet, $2b$ = longueur du coussinet, ϵ, r, h paramètres connus. La solution doit s'annuler pour $\alpha=0$, $\alpha=\pi$, quel que soit z, et pour $z=\pm b$, quel que soit α. En supposant d'après G. Duffing, $p=A_0(\alpha)+\sum A_k(\alpha)Z_k(z)$, on trouve

$$(2)\qquad \frac{d}{d\alpha}\left[(\epsilon-\cos\alpha)^3\frac{dA_0}{d\alpha}\right]+h\sin\alpha=0,$$

$$(3)\qquad \frac{d^2Z_k}{dz^2}=m_k^2Z_k,$$

$$(4)\qquad \frac{d}{d\alpha}\left[(\epsilon-\cos\alpha)^3\frac{dA_k}{d\alpha}+r^2m_k^2(\epsilon-\cos\alpha)^3A_k\right.$$

La résolution de (2) et (3) est facile. Pour trouver la solution approchée de (4) l'auteur pose dans (4)

$$(5)\qquad A_k\simeq(\epsilon^2-1)^{\frac{1}{2}}\sin\alpha[\epsilon-\cos\alpha+a_k(1-\epsilon\cos\alpha)](\epsilon-\cos\alpha)^{-2}$$

et détermine a_k et m_k par la méthode des moindres carrés, en intégrant [le résultat de substitution]^2 de 0 à π. Il montre que la formule approchée (5) donne des meilleurs résultats que d'autres procédés. Le mémoire contient les tables numériques et les graphiques nécessaires dans les applications.

V. A. Kostitzin (Paris).

Squire, H. B. The round laminar jet. Quart. J. Mech. Appl. Math. 4, 321–329 (1951).

An exact axially-symmetric solution of the 3-dimensional incompressible Navier-Stokes equations is found. It is singular at the origin only and is most appropriately interpreted as the flow produced by a continuously applied point force. However, it can also be interpreted with regard to a jet of fluid issuing into a field of the same fluid. The heat transfer problem, introduced by including a point source of heat at the origin, is discussed as is the relation of the laminar to the turbulent jet. *G. F. Carrier* (Providence, R. I.).

Weil, Herschel. On the extrusion of a very viscous liquid. J. Appl. Mech. 18, 267–272 (1951).

The two-dimensional incompressible flow of a liquid from a large reservoir into a narrow channel is considered. The stream function is found and representations appropriate to the channel, to the junction region, and to the "far field", are presented. *G. F. Carrier* (Providence, R. I.).

Kuerti, G. Boundary layer in convergent flow between spiral walls. J. Math. Physics 30, 106–115 (1951).

This is one of the few instances in the theory of viscous fluids in which a rigorous proof has been given for the asymptotic behavior of the solutions of a boundary value problem as viscosity tends to zero. The author considers the class of spiral flows first studied by Hamel [Jber. Deutsch. Math. Verein. 25, 34–60 (1916)], in particular the convergent flows through an arbitrary but fixed spiral channel with adherence at the walls and with the maximum flow speed on each circular arc about the center of the spiral fixed as the viscosity is allowed to vary; it is for this class of flows that the author provides a sound basis for the boundary layer theory. As is well known the Navier-Stokes equations for these flows can be reduced to a nonlinear ordinary differential equation of the second order in which the independent variable does not appear explicitly. An argument based on the topology of the integral curves in the phase plane of this equation shows that it is possible to satisfy the boundary conditions for all sufficiently large values of the Reynolds' number R. Estimates on the slope of the velocity profile for large R then permit the conclusion that as $R\to\infty$ the flows approach uniformly in any closed subregion of the channel the ideal fluid flow having constant flow speed on each circular arc about the spiral center. Using the same estimates more strongly, the author, in analogy with classical boundary layer theory, derives the asymptotic shape of the boundary layer profile and the boundary layer equations corresponding to these flows. *D. Gilbarg.*

Loitsianskii, L. G. Approximate method of integration of laminar boundary layer in incompressible fluid. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1293, 21 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 513–524 (1949); these Rev. 11, 477.

Tamaki, Humio. On the solution of the laminar boundary layer equations. Rep. Inst. Sci. Tech. Univ. Tokyo 5, 49–62 (1951). (Japanese. English summary)

An approximate method for solving the equations for a steady two-dimensional boundary layer by using von Mises' transformation is presented. *From the author's summary.*

Stine, Howard A. The flat plate laminar boundary layer in a steady accelerated compressible fluid. University of Washington. Engineering Experiment Station. Bulletin no. 118, 12–25 (1951).

The boundary layer equations for compressible flow are solved for the pressure proportional to $x^{-\frac{1}{2}}$, x being the distance along the plate. The Prandtl number is taken as .733 and various Mach numbers are considered. The numerical information is given in graphical form.

G. F. Carrier (Providence, R. I.).

Cocchi, Giovanni. Sull'equazione generale del moto nelle correnti turbolente. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 7 (1949–50), 133–142 (1951).

Dowlen, E. M. A comparison of the calculated profile drag coefficients of various low-drag wing sections. Coll. Aeronaut. Cranfield. Rep. no. 35, 10 pp. (25 plates) (1950).

By an approximate method the frictional drag has been calculated for ten low drag aerofoil sections. The main point of the work is apparently to show that when the boundary

layer transition point lies at a fixed chordwise position, rearward movement of the maximum suction point may increase the profile drag coefficient. Inasmuch as this fact is reasonably obvious from a physical point of view, the interest centers about the magnitude of the effect which may amount to 10% increase in an extreme instance. Consequently it is deduced that when the transition point can not be moved further rearward by changing the point of maximum suction in the same direction, the profile drag may be increased if rearward motion of the maximum suction point is continued. It is also shown that since the cusp trailing edge results in low velocities in that region, the skin friction is augmented (in the magnitude of 1.0–1.5%) by modifying the trailing edge to have a finite angle. Some experiments were performed to verify these deductions.

F. C. Marble (Pasadena, Calif.).

***Sauer, R. Écoulements des fluides compressibles.** Librairie Polytechnique Ch. Béranger, Paris et Liége, 1951. xvi+307 pp.

This volume for the most part is a greatly expanded (and improved) version of the author's earlier work, [Theoretische Einführung in die Gasdynamik, Springer, Berlin, 1943; reprinted by Edwards Bros., Ann Arbor, Mich., 1945; these Rev. 7, 92], the two being particularly alike in the choice and presentation of subject matter, and in their mathematical level, which is appropriate to the theoretical engineer and physicist. The principal differences from the earlier work, aside from the much needed expansion of the overly condensed text, are in the sections on unsteady one-dimensional flows (previously omitted) and in the long section of almost one hundred pages on the method of characteristics, where the various combinations of steady plane and axisymmetric flows, and unsteady one-dimensional flows—for both the isentropic and nonisentropic case—are treated in detail by graphical and numerical methods. The sections on shock waves have been amplified to cover more completely the formal details of the shock transition, and to include brief discussions of shock wave interaction and reflection, and of detonation and deflagration waves. For a resumé of the other principal topics, the reader is referred to the review of the work cited above.

The author's prefatory remark that the manuscript for the book was written in 1945–46 seems to account for the pre-1945 vintage of almost all the source material. Nevertheless, this volume serves as a good introduction to the "standard" material around which much of the present day research in the theory of compressible fluids has developed.

D. Gilbarg (Bloomington, Ind.).

***Germain, P. Hypothèses et méthodes générales de l'aérodynamique supersonique linéarisée.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome II. Étude sur la mécanique des fluides. pp. 217–250. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951).

This is a clear expository treatment of the subject. As the title indicates the author confines himself to exposition of the general methods currently in use in the linearized theory of supersonic flow, and makes no applications to particular flow problems. Appended is an extensive and carefully classified bibliography on the subject, including the applications, which should be helpful to anyone trying to find his way in the field.

D. Gilbarg (Bloomington, Ind.).

Chapman, Dean R. Airfoil profiles for minimum pressure drag at supersonic velocities—general analysis with application to linearized supersonic flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2264, 38 pp. (1951).

The problem is to determine the shape of a sharp-nosed symmetrical airfoil moving at Mach number M_∞ at zero angle of attack so that $c_d=2c^{-1}\int_0^c Py'dx-P_bh/c$ is minimized subject to the auxiliary condition $I=c^{-1}\int_0^c(\frac{1}{2}t)^{-\sigma}y^n dx=\text{const.}$ Here c_d, c, t, and h denote the airfoil's section pressure drag coefficient, chord, thickness, and trailing edge thickness; $P(y', y_0', M)$ denotes the local pressure coefficient, the explicit form of which will depend on the type of flow theory employed; y_0' the leading edge slope; and P_b the base pressure coefficient, empirically determined and assumed constant over h. For various combinations of $n=1, 2, 3$ and $\sigma=0, 1$ the constant I is proportional to structural parameters such as torsional stiffness for thin skinned or solid sections, etc. The Euler equation $d(P+y'\partial P/\partial y')/dx+\lambda n y^{n-1}=0$ is subject to boundary conditions $y(0)=0$; $P+(y'\partial P/\partial y')-P_b=0$ at the trailing edge; and $nl/c-\sigma(\frac{1}{2}t)^{\sigma-n}I=0$, where l is the length of the interval on which $y=\frac{1}{2}t$. There is a first integral $y'^2\partial P/\partial y'+\lambda y^n=\text{const.}$ For linearized flow, in which $P=2y'(M_\infty^2-1)^{-\frac{1}{2}}$ this can be integrated explicitly in terms of incomplete elliptic functions or more elementary functions, and $B=-P_b(M_\infty^2-1)^{\frac{1}{2}}c/t$ emerges as the significant parameter. For equal values of I the ratios of drag coefficients of these optimum airfoils (with possibly blunt trailing edges) to those of biconvex circular arc and double wedge airfoils (with sharp trailing edges) have been computed as functions of B and are quite low for the lower range of B values. Earlier work by A. E. Puckett and also by G. Drougge [Flygtekniska Försöksanstalten Meddelande 26 (1949)] was restricted to sharp trailing edges and a somewhat less extensive set of auxiliary conditions.

J. H. Giese (Havre de Grace, Md.).

Ferrari, Carlo. Sul problema del fuso e dell'ogiva di minima resistenza d'onda. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 84, 1–18 (1950).

The author determines the body of revolution, pointed on the axis at both ends, which has minimum pressure drag on the linearised theory of axisymmetric supersonic flow, given that its meridian section passes through a certain point. The approximations to the boundary conditions, under which the calculation has been previously done, were not made. The computation is carried out for a Mach number of two when the thickness at the mid point is one fifth of the length. A similar but slightly less complete investigation is made for the ogive of minimum drag. *M. J. Lighthill.*

Ferrari, Carlo. Sui moti conici rotazionali. Aerotecnica 31, 64–66 (1951).

Some properties of iso-energic rotational flow possessing conical symmetry are worked out. It is pointed out that a conical stream surface (such as a conical solid body) must be a surface of uniform entropy. The same is true of a meridian half-plane of symmetry. Certain conclusions regarding the relationships between surface velocity components can be drawn. Next the problem of continuation, downstream, of the field of velocities from data given on any conical surface is considered. Finally some properties of the hodograph of the velocity in a meridian half-plane are found; these can be used to construct the hodograph from initial data. [In connection with this problem, especially to distribution of entropy in the field, the reviewer

calls attention to a report by A. Ferri [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2236 (1950); these Rev. 12, 875].] *W. R. Sears* (Ithaca, N. Y.).

Hasimoto, Hidenori. On the subsonic flow of a compressible fluid past a Rankine ovoid. J. Phys. Soc. Japan 6, 175–178 (1951).

In Jantzen-Rayleigh's method a solution to the partial differential equation for Stokes' stream function ψ for a steady, irrotational axially-symmetric flow of the form $\psi=\psi_0+\psi_1M^2+\psi_2M^4+\cdots$, M=Mach number of undisturbed flow, is postulated and the functions $\psi_0, \psi_1, \cdots$ of r, θ are required to satisfy

$$D^2\psi_0=0, \quad D^2\psi_1=-\frac{1}{2}\left(\frac{\partial\psi_0}{\partial r}\frac{\partial q_0^2}{\partial r}+\frac{1}{r^2}\frac{\partial\psi_0}{\partial\theta}\frac{\partial q_0^2}{\partial\theta}\right), \cdots.$$

For ψ_0 the author takes the stream function for incompressible flow past a Rankine ovoid and computes a first approximation ψ_1 as a power series in the reciprocal of the radial distance r. The arbitrary constants appearing in the coefficients were adjusted so that $\psi_1(r, \theta)=0$ for $\theta=0, \frac{1}{4}\pi, \frac{1}{2}\pi$ under the assumption that higher order terms may be neglected. The form of the ovoids $\psi_0+M^2\psi_1=0$ is studied together with the distribution of the flow velocity over them. *M. H. Martin* (College Park, Md.).

Pai, S. I. Ring shape source distribution in axially symmetrical supersonic flow. J. Aeronaut. Sci. 18, 634–635 (1951).

Teofilato, Pietro. Determinazione della corrente supersonica tridimensionale col metodo delle caratteristiche. (Generalizzazione del problema). Pont. Acad. Sci. Acta 14, 33–44 (1950). (Italian. Latin summary)

Consider the quasi-linear equation

$$(1) \qquad \sum_1^3 A_{ij}\partial^2\varphi/\partial x_i\partial x_j+\theta=0,$$

where A_{ij} and θ are analytic functions of the x_i and $\partial\varphi/\partial x_i$. Where (1) is of hyperbolic type, the normals μ_i to a characteristic surface S satisfy (2) $\sum A_{ij}\mu_i\mu_j=0$. On S equation (1) and six strip conditions for $\partial\varphi/\partial x_i$, regarded as linear equations for $\partial^2\varphi/\partial x_i\partial x_j$, yield two characteristic (compatibility) conditions which are homogeneous in and involve only tangential derivatives of $\partial\varphi/\partial x_i$. From such conditions the following step-by-step procedure is developed for numerical solution of Cauchy's problem for (1). Suppose $\partial\varphi/\partial x_i$ are known at points M, N, Q. Approximate characteristic surfaces through the segments MN and MQ by two planes whose normals satisfy (2) with A_{ij} evaluated at M. Let P be some point on their intersection. In both characteristic conditions on the plane MNP take the tangential derivatives along MN and MP, approximate them by difference quotients, and evaluate coefficients at M. Proceed similarly with one of the characteristic conditions on MQP to obtain a third linear equation for $\partial\varphi/\partial x_i$ at P. The author fails to suggest that the normals μ_i to MNP and MQP and also the location of P should be chosen carefully to keep P approximately within the domain of determination of M, N, and Q. It should be remarked that characteristic methods for calculation of three-dimensional irrotational supersonic flows have also been described in principle or in detail by C. L. Dolph and N. Coburn [Proc. Symposia Appl. Math., vol. I, pp. 55–66, New York, 1949; these Rev. 10, 751], C. Ferrari [J. Aeronaut. Sci. 16, 411–434 (1949); these Rev. 10, 754], W. E. Moeckel [Tech. Notes Nat. Adv. Comm. Aeronaut. no. 1849 (1949); these Rev. 10, 643], and C. K. Thornhill [Ministry of Supply, Armament Research Establishment, Fort Halstead, Kent, Rep. no. 29/48 (1948); these Rev. 11, 544]. *J. H. Giese* (Havre de Grace, Md.).

Sauer, R. Ausbreitungsgesetze schwacher Verdichtungsstösse in Gasen. Ing.-Arch. 18, 239–241 (1950).

Asymptotic formulas for the decay of weak shock waves have been derived by a number of authors [see, e.g., Friedrichs, Comm. on Appl. Math. 1, 211–245 (1948); these Rev. 10, 638]. In this note the author presents a simple derivation of the asymptotic formula for the decay of a weak shock followed by an expansion in the case of unsteady one-dimensional, cylindrically symmetrical, and spherically symmetrical flows. The derivation is based on the following approximations: (1) the velocity of the shock front is the mean of the sound speeds before and behind the shock front; (2) entropy is constant (in space and time) in the expansion wave behind the shock front; (3) the pressure-density relation in the wave is linear; (4) the wave form is linear, that is, at fixed time t pressure and velocity are linear functions of x from tail to head of the wave. Using a result due to Hantzsche and Wendt [Jahrbuch 1940 der Deutschen Luftfahrtforschung, I536–I538 (1940); these Rev. 9, 217] on the asymptotic formula for the slope of the wave as function of t, the author obtains easily that the excess pressure ratio decreases asymptotically as t^{-1}, t^{-1}, $1/t(\ln t)^{\frac{1}{2}}$ for the respectively one-dimensional, cylindrically symmetrical, and spherically symmetrical flows, and that the length of the wave increases as $t^{\frac{1}{2}}$, $t^{\frac{1}{2}}$ and $(\ln t)^{\frac{1}{2}}$. *D. Gilbarg.*

Kofink, W. Zur Theorie des gegabelten Verdichtungsstosses. Ann. Physik (6) 9, 200–212 (1951).

The problem of three shock configuration without Prandtl-Meyer expansion is formulated by the author as an algebraic equation of 6th degree, assuming constant specific heats, uniform stream before the shocks. The author shows that the degree of the equation can be reduced to first or fourth for various special cases. *H. Tsien.*

Migotsky, E., and Morkovin, M. V. Three-dimensional shock-wave reflections. J. Aeronaut. Sci. 18, 484–489, 504 (1951).

This paper shows that the classical technique for regular shock reflection in two-dimensional supersonic non-viscous flow can be extended to treat the local shock reflection from a three-dimensional body by utilizing the simple geometrical fact that the velocity component locally tangential to the impinging curve of the shock at the body offers no contribution to the shock reflection. The authors call this physical fact the "sweepback principle", because of the analogy to the well-known effect of sweepback in a supersonic wing. The basic method is given both analytically and geometrically. It is also further demonstrated with two examples: (a) a plane shock impinging on an infinite cylinder parallel to the free stream and (b) a conical shock reflected from a plane wall. Just like the two-dimensional case, the regular reflection is restricted to a certain range of deflection angle for a given effective Mach number which is, of course, smaller than the incoming Mach number. (This restricted range corresponds to the velocity component being normal locally to the impinging line of the shock at the body.)

Furthermore, in the case of a cylindrical body there is a region on the lee side of the body, where the regular reflection cannot exist. Instead, an expansion wave exists to turn

the flow back along the cylinder. The implication of the boundary layer effect in actual flow is briefly indicated. A misprint is found on page 485, line 17 (bottom), first word: "2" should be changed to "6" (confirmed by one author).

C. C. Chang (Baltimore, Md.).

Stocker, P. M. On a problem of interaction of plane waves of finite amplitude involving retardation of shock-formation by an expansion wave. Quart. J. Mech. Appl. Math. **4**, 170–181 (1951).

This paper studies by means of explicit solutions the influence of an expansion wave in retarding shock formation in a compression wave. In the unsteady, one-dimensional, isentropic flow of a perfect gas with $\gamma=5/3$, the author calculates by Riemann's method of characteristic coordinates the time and place of the initial appearance of a limit line in the (x, t)-plane when a receding simple compression wave interacts with an advancing simple expansion wave, and compares these quantities with those corresponding to the compression alone. He finds, in the case of a compression wave generated by a parabolic piston curve in the (x, t)-plane, that the time and place of formation of the limit line can be delayed by factors of as much as 3.73 and 9.70, respectively.

D. Gilbarg (Bloomington, Ind.).

Stocker, P. M., and Meyer, R. E. A note on the correspondence between the x, t-plane and the characteristic plane in a problem of interaction of plane waves of finite amplitude. Proc. Cambridge Philos. Soc. **47**, 518–527 (1951).

A study of the singularities in the mapping between the (x, t)-plane and the plane of the Riemann characteristic variables when a receding simple compression wave interacts with an advancing simple expansion wave. For these flows the image of the characteristic plane in the (x, t)-plane is four-sheeted, the four sheets being joined pairwise along two limit lines which meet in a point where each is cusped. This paper provides specific examples of singularities which had been studied in general terms in connection with two-dimensional steady flows by Craggs [Proc. Cambridge Philos. Soc. **44**, 360–379 (1948); these Rev. **10**, 640] and Meyer [Philos. Trans. Roy. Soc. London. Ser. A. **242**, 153–171 (1949); these Rev. **11**, 473]. *D. Gilbarg.*

Krzywoblocki, M. Z. On the limiting lines in inviscid rotational and viscous flows. Bull. Calcutta Math. Soc. **42**, 153–171 (1950).

In the present paper an attempt is presented to solve the problem of the existence or non-existence of a limiting line in a viscous fluid. The attack is based on the extended Tollmien's theory. The considerations, given below, refer parallelly to an inviscid rotational and to a viscous flow. It will be shown that under certain conditions less restrictive for an inviscid fluid and more restrictive for a viscous fluid, both types of flow may be treated simultaneously. The author did not attempt to present the way of construction of the solution and of the intermediary items (like characteristics, Mach lines, limiting lines, etc.). Only the proof concerning the existence theorem will be presented. This approach in the case of a viscous fluid enables one to use the characteristics of the reduced order equation (second order) instead of the singularities of the original equation (third order). Thus, although one is not able to find the singularities in a viscous fluid vector field of flow, one may extend the considerations referring to a limiting line to this type of flow. A brief representation of the hypothesis of a shock in viscous gases closes the paper. (Author's summary.)

M. J. Lighthill (Manchester).

Oswatitsch, Klaus. Ähnlichkeitsgesetze für Hyperschallströmung. Z. Angew. Math. Physik **2**, 249–264 (1951).

The concept of hypersonic similarity and the hypersonic similarity law were given independently by the reviewer [J. Math. Physics **25**, 247–251 (1946); these Rev. **8**, 237] and by S. V. Falkovich [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 459–464 (1947); translated as Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1239** (1949); these Rev. **9**, 476; **11**, 272] and later extended by W. D. Hayes [Quart. Appl. Math. **5**, 105–106 (1947); these Rev. **8**, 610] to include the nose-shock effects. These authors define hypersonic flow as flows over slender or thin bodies where the product K of the Mach number and thickness ratio of body is of the order of unity or larger than unity. The present author limits himself to the case $K\to\infty$. The original concept of hypersonic similarity is thus more general than that of the present author. Through a misunderstanding on the author's part, he states that the Tsien-Falkovich hypersonic similarity law is invalid. This is, of course, not true since the agreement of the predictions of this similarity law with both exact calculations [S. E. Neice and D. M. Ehret, J. Aeronaut. Sci. **18**, 527–530 (1951)] and experimental data [C. H. McLellan, ibid. **18**, 641–648 (1951)] is excellent. The author did make an important contribution in this paper, however: He definitely showed that for a fixed body configuration, the flow pattern is frozen, independent of the free stream Mach number M_∞, when $M_\infty\to\infty$. This means that the pressure coefficient over the surfaces of the body is independent of M_∞ as $M_\infty\to\infty$. *H. S. Tsien* (Pasadena, Calif.).

Mark, Richard M. Application of the extended Kármán-Tsien method for the generation of conventional airfoils in two-dimensional subsonic compressible flow. University of Washington. Engineering Experiment Station. Bulletin no. **118**, 7–11 (1951).

A method is developed for generating a class of simple airfoils (the inverse problem) in a steady, two-dimensional, non-viscous, compressible flow subject to the linearized pressure-density relation. This method is a special case of the extension by Lin of the procedure of Kármán and Tsien and involves a supplementary condition also due to Lin [Note on the extended von Kármán-Tsien method, unpublished paper] which ensures that the airfoil be a conventional airfoil. A numerical example is given.

C. Saltzer (Cleveland, Ohio).

Scholz, N. On an extension of Glauert's theory of thin airfoils to profiles in cascade. J. Aeronaut. Sci. **18**, 637–639 (1951).

Loitsianskii, L. G. Generalization of Joukowski formula to an airfoil of a cascade in compressible gas stream with subsonic velocities. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1304**, 16 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 209–216 (1949); these Rev. **11**, 276.

Loitsianskii, L. G. Resistance of cascade of airfoils in gas stream at subsonic velocity. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1303**, 30 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 171–186 (1949); these Rev. **11**, 276.

Eula, A. Caratteristiche aerodinamiche di ali a freccia con bordo d'attacco subsonico e bordo d'uscita supersonico. I, II, III. Aerotecnica **30**, 107–113, 175–182 (1950); **31**, 103–109 (1951).

In Part I linearized supersonic conical flow methods have been applied to calculate the lift, induced drag, pitching, and bending moment coefficients of a non-yawing symmetrical plane wing of zero thickness, whose plan form is the sum or difference of two isosceles triangles with a common base, and whose leading (trailing) edges lie inside (outside) the Mach cone of their intersections. The work closely resembles that of A. E. Puckett and H. J. Stewart [J. Aeronaut. Sci. **14**, 567–578 (1947)]. Part II extends the computations to symmetrical wings produced by replacing the trailing edge of each half by two straight segments, both outside the Mach cones of their intersection. Part III is an exposition of the work of A. E. Puckett [J. Aeronaut. Sci. **13**, 475–484 (1946); these Rev. **8**, 109] on the drag of thin delta wings at zero incidence. Each part contains numerous graphs to facilitate computations of aerodynamic coefficients as functions of Mach number and various geometrical parameters. *J. H. Giese* (Havre de Grace).

Reissner, Eric. On the application of Mathieu functions in the theory of subsonic compressible flow past oscillating airfoils. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2363**, 42 pp. (1951).

The author sets forth his earlier analysis [with S. Sherman, Curtiss-Wright Corp. Rep. **S.B. 240-S-1** (1944)] for the two-dimensional problem, modifying the treatment of that portion of the solution required to satisfy the Kutta condition along lines proposed by Haskind [Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 129–146 (1947); these Rev. **9**, 478]. He then shows how three-dimensional corrections may be incorporated. While the method has the advantage, relative to other treatments [Karp, Shu, and Weil, Tech. Rep. no. **F-TR-1167-ND** (1947), Air Materiel Command, Wright Field, Dayton, Ohio; these Rev. **9**, 632], of exhibiting the solution in closed form, a considerable amount of numerical work will be required before the theory can be applied. *J. W. Miles* (Los Angeles, Calif.).

Cicala, P. Present state of development in nonsteady motion of a lifting surface. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1277**, 96 pp. (1951).

Translated from Aerotecnica **21**, 557–591, 671–685, 759–766, 766–773 (1941).

Nelson, Herbert C. Lift and moment on oscillating triangular and related wings with supersonic edges. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2494**, 34 pp. (1951).

Goodman, Theodore R. Aerodynamics of a supersonic rectangular wing striking a sharp-edged gust. J. Aeronaut. Sci. **18**, 519–526 (1951).

Gardner's method [Comm. Pure Appl. Math. **3**, 33–43 (1950); these Rev. **12**, 62] is applied to the title problem to obtain the pressure distribution, the integral for which is intractable within the zone of action of the intersection of gust front and wing edge. The spanwise integrals of the pressure are shown to be in agreement with those previously obtained by the reviewer [J. Aeronaut. Sci. **17**, 647–652 (1950); these Rev. **12**, 369]. The author remarks that pressure distribution (p) is homogeneous in x, y, ct (x and y streamwise and spanwise coordinates, c sonic velocity and t time) and exhibits $p(x/ct, y/ct)$ in an instructive, isometric drawing. *J. W. Miles* (Los Angeles, Calif.).

Campbell, John P., and McKinney, Marion O. Summary of methods for calculating dynamic lateral stability and response and for estimating lateral stability derivatives. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2409**, ii+97 pp. (1951).

Garfinkel, Boris. Minimal problems in airplane performance. Quart. Appl. Math. **9**, 149–162 (1951).

Neglecting the small inertial effects of normal airplane operations, the author considers the performance of a propeller-driven airplane as a problem of a particle in equilibrium under the forces of drag, lift, thrust and gravity. The relation between lift and drag is specified through the lift polar. The engine is considered to be fully supercharged so that the range of available power is independent of the altitude. The specific fuel consumption is assumed to be a function of power only. Curiously, however, the author also assumes the propulsive efficiency of the propeller to be only a slowly varying function of air density, independent of speed of the airplane. This is not the experimental fact. The efficiency of propeller is a rapidly varying function of speed. In the reviewer's opinion, this single unfortunate assumption will greatly limit the usefulness of any result coming from the author's theory. In the absence of lateral wind, the general flight plan is considered as an arc in the four-dimensional space of time, (x, y)-coordinates and the mass of the airplane, varying due to the consumption of fuel. The author solves the problem of minimizing an arbitrary function of the end-values in this generalized coordinates. This variational problem is first reduced to the problem of Bolza. The Euler-Lagrange equations then give a two-parameter family of curves, obtainable for any particular airplane by using machine computers. The complications arising from the presence of inequalities and a discontinuity can be resolved by the so-called "corner condition". The process of selecting from the family the curve that solves a particular problem is carried out by scanning the end-conditions and the transversality condition at the end point. *H. Tsien.*

***Carrière, P. Méthodes théoriques de la balistique.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome II. Étude sur la mécanique des fluides. pp. 7–36. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951).

Expository paper.

Ščelkačev, V. N. The application of operational methods to the solution of a problem of the motion of an elastic liquid in an elastic stratum. Doklady Akad. Nauk SSSR (N.S.) **79**, 751–754 (1951). (Russian)

An elastic liquid is assumed to be moving in an elastic porous stratum toward the axis of a circular cylindrical pit sunk into the stratum. The regime within the stratum is elasto-hydrostatic. Around the pit there is assumed to be a ring-shaped region in which the coefficients of porosity, permeability, piezoconductivity and elasticity are constants different from the constant values which the respective coefficients have in the exterior of this region. The problem is to determine the pressure $p(r, t)$ in the stratum at any time t (after the pit has been sunk) and at any distance r from the axis of the pit. The initial pressure is assumed to be the same at all points. This problem has its analogue in the theory of heat conduction. It is solved here by the use of the Laplace transform. *H. P. Thielman.*

Polubarinova-Kočina, P. Ya. On the theory of unsteady motions in a many layered medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 511–514 (1951). (Russian)

In this paper the author studies a function $\Phi(x, y)$ which can be considered as a generalized potential averaged over the height in case the coefficient of filtration changes with the height. This function has the form

$$\Phi(x, y) = \int_0^h (z-h)k(z)dz,$$

where h is the pressure, considered as a function of the rectangular coordinates x, y in a vertical plane perpendicular to the layers of the medium through which the filtration takes place, and k is the coefficient of filtration which changes with the height. The equations which Φ must satisfy for the case of a two-layer medium are determined and special solutions of these equations are compared with the solutions for the case of the single-layer problem.

H. P. Thielman (Ames, Iowa).

Munk, W. H., Groves, G. W., and Carrier, G. F. Note on the dynamics of the Gulf Stream. J. Marine Research **9**, 218–238 (1950).

On se trouve actuellement en présence de deux théories de la circulation océanique avec l'action du vent: celle de Rossby (bien connue sous le nom de "Jet Stream") et celle de Stommel reprise et développée récemment par W. H. Munk [J. Meteorol. **7**, 79–93 (1950)]. Une différence fondamentale entre ces théories consiste en ceci, que la théorie de Stommel et la théorie linéaire de Munk négligent les termes d'inertie, tandis que dans la théorie de Rossby ces termes sont prépondérants. Afin d'expliquer les particularités de la circulation du Gulf Stream, les auteurs essaient de réintroduire dans leur théorie les termes d'inertie en partant de la solution sans termes d'inertie obtenue précédemment et en opérant par approximations successives. Malheureusement, le résultat pratique n'est pas bien concluant; il faut, peut-être, considérer l'action d'autres causes dynamiques et thermodynamiques. *M. Kiveliovitch* (Paris).

McVittie, G. C. Coordinate systems in dynamic meteorology. J. Meteorol. **8**, 161–167 (1951).

This paper is concerned with the transformation of the equations of hydrodynamics to various curvilinear coordinate systems convenient for meteorology, the earth (without rotation) being regarded as a Newtonian frame of reference. In addition to several purely geometrical coordinate systems, the author considers intrinsic orthogonal coordinates in which one of the coordinates at a point P is a physical quantity such as pressure, measured at the point on the earth's surface vertically below P. The method of transforming coordinates is the tensorial method of the author [Proc. Roy. Soc. London. Ser. A. **196**, 285–300 (1949); these Rev. **10**, 633]. The paper includes a brief discussion of the relationship between the size of an atmospheric motion and the coordinate system which best describes it, gradient and geostrophic winds, and the Rossby formula for potential-vorticity. *J. L. Synge* (Dublin).

Wilkes, M. V. The thermal excitation of atmospheric oscillations. Proc. Roy. Soc. London. Ser. A. **207**, 358–370 (1951).

The linearised equations governing non-adiabatic oscillations of the atmosphere on a spherical rotating Earth are set up. These equations are applied to the discussion of the oscillations excited in the atmosphere when it is periodically heated and cooled near the ground. It is shown that these oscillations are the same as those which would be excited (except near the ground) by a certain gravitational potential. The results are obtained for an arbitrary variation of temperature with height but the case of an atmosphere heated by conduction from the ground is also considered. Some numerical examples serve to sustain Chapman's hypothesis that the thermal and gravitational oscillations of the atmosphere are of the same order of magnitude.

G. C. McVittie (London).

Jeffreys, Harold. The surface elevation in cellular convection. Quart. J. Mech. Appl. Math. **4**, 283–288 (1951).

In his experiments with convection cells with a free upper surface, Bénard found that the upper surface of the cells was convex downwards. Since Bénard's explanation of this effect is not correct, an exact solution of the problem is carried through. The result indicates that the upper surface should be convex upward, in contradiction to the experiment. Since errors of sign are quite likely, the result was checked by two entirely different methods. The first method was one of finite differences, which converged too slowly to the desired result. The second consisted in the assumption of a polynomial for the solution which fitted the boundary conditions accurately. The method was satisfactory in that it confirmed the sign of the first result. Hence the disagreement with observations still remains. *H. Panofsky.*

Haskell, N. A. Asymptotic approximation for the normal modes in sound channel wave propagation. J. Appl. Phys. **22**, 157–168 (1951).

The author considers wave propagation in a medium in which the velocity is a continuously varying function of one coordinate. Normal mode solutions of the wave equation are given and discussed. A special form is then assumed for the velocity-altitude function. For this special function the normal modes are approximated. After an approximate calculation of the roots corresponding to these modes, further approximations for the expressions of the velocity potential are derived on this basis. The normal modes are interpreted geometrically and the corresponding phase and group velocities are considered. Integral expressions are derived for the normal modes and these are then approximately evaluated. These approximations are illustrated graphically.

M. J. O. Strutt (Zurich).

Elasticity, Plasticity

Volterra, Enrico. On elastic continua with hereditary characteristics. J. Appl. Mech. **18**, 273–279 (1951).

The stress-strain relation for an elastic material exhibiting linear heredity is discussed. The integro-differential equation for oscillation of a single degree of freedom system is obtained, and transformed into a differential equation when the heredity function is the sum of exponentials. The general equations of motion are developed for an elastic continuum with hereditary characteristics for both volume and shear distortion. Application is made to radial free vibrations of an isotropic sphere, and lateral vibrations of strings, bars, and membranes. In all cases space and time variables separate, and the latter lead to ordinary differential equations when heredity functions in the form of sums of exponentials are assumed. *E. H. Lee* (Providence, R. I.).

Leutert, Werner. The heavy sphere supported by a concentrated force. Pacific J. Math. 1, 97–101 (1951).

A displacement function in closed form, from which displacements can be obtained by differentiation, is given for a heavy isotropic homogeneous elastic sphere supported by a point force at the lowest point. The solution was obtained by taking a Boussinesq type displacement function which satisfies the elasticity equations with a gravity body force, and adding combinations and derivations of known displacement functions for bodies extending to infinity, to satisfy the boundary conditions on the spherical surface.

E. H. Lee (Providence, R. I.).

Fogagnolo Massaglia, Bruna. Sul moto di una sfera elastica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 84, 19–30 (1950).

R. Einaudi [same Atti 74, 591–618 (1939); these Rev. 1, 316] solved the elastokinetic problem of a sphere for which for $t=0$ the displacement and velocity of every point of the sphere's mass and for $t\geqq 0$ the displacement of all points of its surface are given. In the present paper the methods developed by Einaudi are modified for the case when for $t\geqq 0$ the surface tractions, rather than the surface displacements, are given. *P. Neményi* (Washington, D. C.).

Montag, H. Die unendlich ausgedehnte Scheibe mit gleichförmig am Rand belastetem quadratischen Loch. Ing.-Arch. 19, 155–161 (1951).

Th. Schade showed [Ing.-Arch. 19, 118–127 (1951); these Rev. 13, 245] how certain boundary value problems for rectangular regions can be solved in terms of Grammel's generalized trigonometric functions. In the present paper this is done for the plane state of strain of an infinite solid with a square prismatic hole the surface of which is under uniform pressure. The author shows that the computations are simplified if the new variable $V=(1+v^2)^{-1}$ is introduced. The results are evaluated numerically and the stress field represented by aid of the stress-trajectory net and other diagrams. *P. F. Neményi* (Washington, D. C.).

Weber, C. Allseitig gezogene Ebene mit Zweibogenloch. Z. Angew. Math. Mech. 31, 193–201 (1951). (German. English, French, and Russian summaries)

This paper deals with the plane stress problem of an infinite isotropic plate under all-round tension at infinity when the plate contains a hole which is bounded by two arcs of circles. Such a hole involves, of course, a sharp corner in the plate where we expect the stress to be very large. Using the process of inversion the region of the plate outside the hole is transformed to a sector of the complete plane subtending an angle 2α at the origin. Because of the symmetry of the conditions in the plate the conditions along the circular arc of unit radius in this sector can be determined. A general solution for the Airy stress function for this sector in which the straight edges, corresponding to the edge of the hole, are free from traction is determined as a sum of particular solutions multiplied by arbitrary constants. By limiting the number of such particular solutions so that the arbitrary constants are thereby limited, these same constants may be determined from the conditions on the circular arc of unit radius. An approximate solution for the Airy stress function for the original plate is thereby deduced. A numerical result is given when the angle at the inner corner of the hole is $\frac{1}{3}\pi$ and it is shown that the solution yields a good approximation. *R. M. Morris* (Cardiff).

Teissier du Cros, François. Sur l'expression au moyen de deux fonctions de variable complexe des déplacements dans un prisme à connexion multiple où règnent des tensions résiduelles. C. R. Acad. Sci. Paris 233, 127–129 (1951).

In the first part of this paper, the author derives the expression, in terms of two arbitrary complex functions, for the complex displacement $w=u+iv$, in plane strain or generalised plane stress in an isotropic material in a simply connected domain under no body forces. He also expresses the components of stress and the Airy stress function in terms of these two arbitrary functions. These results have previously been given by Stevenson [Proc. Roy. Soc. London. Ser. A. 184, 129–179 (1945); these Rev. 8, 115] in the more general case when body force is included. The author then goes on to extend the result to multiply connected regions which involve singular points and internal boundaries. He gives a general expression for the result but does not show why it should take the particular form he gives, except that he indicates that there is a dislocation between the boundaries. *R. M. Morris* (Cardiff).

Teissier du Cros, François. Sur la décomposition d'un état d'équilibre quelconque en deux états simples qui dérivent d'une fonction holomorphe chacun. C. R. Acad. Sci. Paris 233, 223–225 (1951).

In this paper the author shows that any state of generalised plane stress or plane strain can be resolved into two simple states each of which can be derived from single holomorphic functions. He then gives some of the properties of these two simple states. *R. M. Morris* (Cardiff).

Teissier du Cros, François. Sur certaines intégrales de contour invariantes dans une éprouvette de photoélasticimétrie. C. R. Acad. Sci. Paris 233, 282–283 (1951).

Using the results for the components of the stress in a simply connected region under plane strain or generalised plane stress established in the first paper [see the second preceding review], the author first shows that the gradient of the complex stress function $M=\frac{1}{2}(N_1-N_2-2iT_3)$ at any point z_1 interior to the region can be determined in terms of the integral of M round the circle centre z_1. He then establishes the invariant nature of certain other integrals involving M and its gradient, for contours in the region.

R. M. Morris (Cardiff).

Vlasov, V. S. Basic differential equations in general theory of elastic shells. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1241, 58 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 8, 109–140 (1944); these Rev. 7, 42.

Tsuboi, Yoshikatsu. Fundamental formulas for thin shells. J. Jap. Soc. Appl. Mech. 4, 67–72 (1951). (Japanese. English summary)

In this paper formulas giving the relation between the stresses and deformations of thin shells are established using curvilinear coordinates and are compared with Trefftz's expressions which were obtained by Castigliano's principle.

From the author's summary.

Zerna, W. Membrantheorie verallgemeinerter Rotationsschalen. Ing.-Arch. 19, 228–230 (1951).

The author introduces an interesting new class of shells, which he calls "generalized shells of revolution." Let $x(\theta)$, $y(\theta)$ be parametric equations of a fixed curve in the (x, y)-

plane. Let $\xi(\phi)$, $\eta(\phi)$ be parametric equations of a fixed curve in the (x, y)-plane, and let $r(\theta)$ be a given function. Then the author's surfaces are of the type

$$\mathbf{r}=x(\theta)\mathbf{i}+y(\theta)\mathbf{j}+r(\theta)[\xi(\phi)\mathbf{k}+\eta(\phi)\mathbf{j}].$$

Included as special cases are surfaces of revolution and surfaces of translation. The author derives the statical equations for a shell of this form for the extensional theory, in which bending and twisting moments are neglected. The result includes as a special case the author's previous formulation of the membrane theory of shells of revolution [Ing.-Arch. 17, 223–232 (1949); these Rev. 11, 558] .He is able to introduce a stress function Φ, which is proportional to the stress-resultant N_θ and from which both N_ϕ and $\partial N_{\theta\phi}/\partial\phi$ are immediately determined, which satisfies the equation

$$\frac{\partial^2\Phi}{\partial\theta^2}+G\frac{\partial^2\Phi}{\partial\phi^2}+\left(\frac{\partial G}{\partial\phi}+F\right)\frac{\partial\Phi}{\partial\phi}+J\Phi=q,$$

where F and G are determined by the form of the surface and q by the load distribution. He remarks that if N_ϕ or N_θ be prescribed on the boundary, then boundary values for Φ are given. *C. Truesdell* (Bloomington, Ind.).

Marguerre, K. Stability of the cylindrical shell of variable curvature. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1302, 64 pp. (1951).

The problem of the title is important in the case of the analysis of the leading edge skin of aircraft. The stability of such shells depends greatly on the manner in which the curvature varies. Assuming that the curvature of the shell is small, the author solves the stability equations

$$\Delta\Delta\phi-\frac{E}{R}\omega_{xx}=0,\quad \frac{Et^2}{12(1-\nu^2)}\Delta\Delta\omega+\frac{1}{R}\phi_{xx}=-\sigma\omega_{xx}+2\tau\omega_{xy},$$

where ϕ is the stress function giving the additional membrane stresses caused by buckling, $N_x/t=\partial^2\phi/\partial s^2$, $N_{xs}/t=-\partial^2\phi/\partial x\partial s$, $N_s/t=\partial^2\phi/\partial x^2$, σ is the critical compressive stress, τ the critical shearing stress, x the direction along the shell, and s the direction around the shell. It is assumed to be possible to write the curvature $1/R$ as a few terms of a Fourier expansion in s. A general theory is given and details worked out for a symmetrical half oval shell under (a) axial compression, (b) torsional shearing and (c) axial compression plus torsional shearing. *H. D. Conway.*

Goldenveizer, A. L. Theory of thin-walled rods. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1322, 53 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 561–596 (1949); these Rev. 12, 142.

Dzhanelidze, G. Y. On the theory of thin and thin-walled rods. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1309, 18 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 597–608 (1949); these Rev. 12, 652.

Šerman, D. I. On the stresses in a twisted circular beam weakened by a prismatic cavity. Izvestiya Akad. Nauk SSSR Otd. Tehn. Nauk 1951, 969–995 (1951). (Russian)

Saint Venant's torsion problem for a long circular beam weakened by a rectangular prismatic cavity with rounded corners is solved by the introduction of an auxiliary function which assumes on the circular boundary the same values as the complex torsion function. The auxiliary function is shown to satisfy a Fredholm integral equation whose kernel can be replaced, with a known degree of approximation, by a degenerate kernel. Thus the determination of the auxiliary function is reduced to the solution of a system of linear algebraic equations. Once the auxiliary function is known, the torsion function can be computed. Although the convergence of the approximating process is not fully established in the paper, extensive numerical computations testify to the remarkable effectiveness of the proposed method, even when it is applied to a thin-walled section.

I. S. Sokolnikoff (Los Angeles, Calif.).

Cheng, Che-Min. Restricted torsion of thin-walled columns of air-foil sections. Eng. Rep. Nat. Tsing Hua Univ. 3, no. 1, 80–102 (1947). (English. Chinese summary)

An investigation is given of the torsion of a thin-walled cylinder, stiffened internally by a system of ribs that prevents distortion of normal cross-sections only in their own plane, when axial warping of certain normal cross-sections is prevented. The analysis elaborates that presented previously by von Kármán and W. Z. Chien [J. Aeronaut. Sci. 13, 503–510 (1946); these Rev. 8, 119]. A numerical example is discussed in detail. *H. G. Hopkins* (Manchester).

Lin, T. C., and Whitehead, L. G. The St. Venant torsion problem for the hyperbolic airfoil cross section. University of Washington. Engineering Experiment Station. Bulletin no. 118, 108–111 (1951).

This paper is an abstract from a Ph.D. thesis written by one of the authors and deals with the determination of the torsion function for a hyperbolic airfoil, being the inverse of a hyperbola. The internal conformal transformation of the hyperbolic airfoil to a unit circle is established and using Schwarz's integral formula, the torsion function is found as a function of the variable s which is the complex variable for the fourth quadrant in the s-plane, this fourth quadrant being transformed to the inside of the unit circle in the t-plane by the transformation $t=[(s+i)/(s-i)]^2$. No expressions for the stresses or displacements are determined but the authors say that the displacements and shearing stresses along the boundary were determined in the original thesis, as well as the torsional rigidity. Since the displacements and stresses are not determined it is not shown that these are single-valued and continuous functions at all points within the boundary. By varying the constants involved in the transformation the torsion function for various airfoils with different tail angles, camber, and chord length may be determined. *R. M. Morris* (Cardiff).

Lin, T. C., and Yang, H. T. The St. Venant torsion problem for cross sections consisting of one loop of the hyperbolic limaçon. University of Washington. Engineering Experiment Station. Bulletin no. 118, 112–119 (1951).

Following a method given in a previous paper [reviewed above] by one of the authors, an exact solution is found of the St. Venant torsion problem for a cylinder the cross section of which is one loop of the hyperbolic limaçon with tail angle equal to π/k (k being a positive integer). Expressions are found for the resultant shearing stress on the boundary and also the torsional rigidity, and these are evaluated in the particular cases when $k=1$, when the airfoil is a circle, and when $k=2$, when the profile of the airfoil is symmetrical with tail angle $\pi/2$. *R. M. Morris* (Cardiff).

Abramyan, B. L., and Džrbašyan, M. M. On the torsion of shafts of variable cross-section. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 451–472 (1951). (Russian)

A solid cylindrical shaft consisting of two circular cylinders of different radii is twisted by the axially symmetric forces distributed on the lateral surface of the shaft. An exact solution of this torsion problem is given for the case when the external load is a function of bounded variation. The stress function is represented in a series of Bessel's functions whose coefficients are determined from a completely regular system of linear algebraic equations. Formulas for stresses, depending on the geometrical parameters of the shaft, are given. *I. S. Sokolnikoff.*

Rosenberg, R. M. The influence of axial torques on the critical speeds of uniform shafts in self-aligning bearings. University of Washington. Engineering Experiment Station. Bulletin no. 118, 88–94 (1951).

Employing a rectangular set of coordinates which rotates with the shaft and has its origin at the midpoint of the span, the author constructs the kinetic and potential energy functions of a vibrating beam under the action of both bending and axially applied torque. The vanishing of the variational derivative of this total energy leads to an eigenvalue problem. Symmetry properties in the motion of the shaft and in its end conditions reduce the difficulties of handling the determinantal equation. The problem is solved in detail for a shaft rotating in self-aligning end bearings. It is found that the critical speeds decrease with increasing axial torque and is most pronounced in the lowest mode. In each mode there is a critical torque sufficient to cause instability, that is to reduce the corresponding speed to zero. *D. L. Holl* (Ames, Iowa).

Dörr, J. Bestimmung der Dreheigenfrequenzen einer gewissen Gruppe von Wellen mit singulären Rändern. Österreich. Ing.-Arch. 5, 217–225 (1951).

Torsional modes of vibration of tapered beams having straight axes are calculated when the torsional stiffness is given by $g(x)=K[\cos(\frac{1}{2}x)]^{p_1}[\sin(\frac{1}{2}x)]^{p_2}$ where x is a length variable normalized so that the ends are at $x=0$ and $x=\pi$. Further restrictions are: $p_1\leqq 0$, $p_2\leqq 0$ for the fixed-fixed beam; $p_1>-\frac{1}{2}$, $p_2>-\frac{1}{2}$ for the free-free beam; $p_1\leqq 0$, $p_2>-\frac{1}{2}$ for the fixed-free beam ($x=0$ end free). The theory of the Riemann P-equation is employed, and vibration frequencies are obtained in simple closed form. *E. Pinney* (Berkeley, Calif.).

Eringen, A. Cemal. On the non-linear vibration of elastic bars. Quart. Appl. Math. 9, 361–369 (1952).

The lateral vibrations of a bar whose ends cannot translate are investigated. The non-linear equations are treated by using a perturbation series in a geometric parameter of the bar. The results appear as elliptic functions in the time. *G. F. Carrier* (Providence, R. I.).

Kellis-Borok, V. I. On surface waves in a layer resting on an elastic semispace. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1951, no. 2, 17–39 (1951). (Russian)

L'auteur étudie quelques propriétés des ondes superficielles stationnaires produites par l'action d'une source ou une perturbation ponctuelle sinusoïdale dans une couche liée de façon rigide à un demi-espace élastique sous-jacent. La couche et le demi-espace sont supposés homogènes, isotropes et idéalement élastiques. L'intensité, la dispersion, l'interférence de ces ondes sont étudiées dans le cas de deux et de trois dimensions ainsi que le comportement des racines de l'équation de fréquences, en considérant la couche en question comme image moyenne de l'écorce terrestre. Les résultats théoriques sont confrontés avec le matériel expérimental. Il parait que les phénomènes seraient en contradiction avec la théorie classique simplifiée et qu'il y aurait passages multiples (continus ?) entre les ondes superficielles étudiées par l'auteur et celles de Rayleigh et de Love. *A. Kostitzin* (Paris).

Yoffe, Elizabeth H. The moving Griffith crack. Philos. Mag. (7) 42, 739–750 (1951).

The stress distribution near the apex of a narrow plane crack moving parallel to itself is sought for an elastic plate with tension normal to the crack. The problem is replaced by considering a moving narrow slit $ct-a<x<ct+a$, $y=0$, since the ends have little mutual influence. This can be considered as a problem in the domain $y>0$, with boundary conditions on $y=0$, and at infinity, which is solved by a superposition and integration of known solutions for surface waves, and an interpretation in terms of complex variables and a development by analogy with the known static solution. As the wave velocity c approaches zero the solution is shown to approach the known static case. Application to the propagation of a fracture crack is made, and the solution is shown to imply deviation of the crack from its plane at high speeds of propagation. *E. H. Lee.*

Chadaya, F. G. Investigation of the stability of a rectangular plate of variable thickness by the method of finite differences. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 191–201 (1949). (Russian. Georgian summary)

S. E. Mikeladze [e.g., Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 219–222 (1948); these Rev. 9, 622] has developed a general method of solving by finite differences the differential equations with boundary conditions of applied mechanics. The author used the above mentioned method for finding the coefficient of stability of a thin rectangular plate hinged on the edges. The plane of symmetry of the plate passes through the (x, y)-plane and two edges coincide with the x and y axes respectively. The variable thickness of the plate is a function of y and the plate is compressed along the edges parallel to x-axis. The author compares his results with those for a square plate with constant rigidity, given by S. P. Timoshenko [Theory of Elastic Stability, McGraw Hill, New York, 1936]. The agreement is good. The author also claims that the method of finite differences is simpler and requires less computations than the one shown by P. F. Papkovich [Applied Mechanics of a Ship, vol. II, Gosstrolizdat, Leningrad, 1941 (Russian)]. *T. Leser.*

Woinowsky-Krieger, S. Über die Beulsicherheit von Rechteckplatten mit querverschieblichen Rändern. Ing.-Arch. 19, 200–207 (1951).

The edges $y=\pm\frac{1}{2}b$ of a rectangular plate are subjected to uniform compressive loads and are not restrained from moving in a direction perpendicular to the plane of the plate. The remaining edges are simply supported. The boundary conditions on the loaded edges are: (1) the bending moments vanish; (2) the usual expression for the reaction of the supports is equal to the component of the load in a direction normal to the deflected plate. Two other cases are considered in which one of the loaded edges is either simply supported or clamped and the other is free to move in a direction perpendicular to the plane of the plate. *H. W. March.*

Krall, Giulio. Stabilità dell'equilibrio elastico. Ann. Mat. Pura Appl. (4) **29**, 75–90 (1949).

Various known criteria used in the analysis of the instability of elastic systems are developed from the fundamental principle of Dirichlet on the equilibrium of conservative systems. Certain specific examples of elastic stability problems are discussed in detail. *H. G. Hopkins.*

Handelman, G. H., and Prager, W. Plastic buckling of a rectangular plate under edge thrusts. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **946**, 28 pp. (1949).

First issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1530** (1948); these Rev. **10**, 82.

Green, A. P. A theoretical investigation of the compression of a ductile material between smooth flat dies. Philos. Mag. (7) **42**, 900–918 (1 plate) (1951).

Stress and velocity distributions are developed for an ideal plastic-rigid material pressed symmetrically between plane smooth dies in plane strain for any width-to-height ratio greater than one. The method of solution is based on a transformation of the velocity field into the slip-line field. Integration for the velocity field cannot be carried through directly since boundary values are prescribed on only one of an initial pair of characteristics. Superposition is therefore used, and influence coefficients calculated for unit boundary value problems. For integral ratios of width to height a constant die pressure equal to the yield stress in plane strain is deduced, for other ratios a higher average pressure is obtained. Simple analytic relations between velocity and pressure parameters are given, the accuracy of agreement suggesting that they are exact. The average pressure is compared with upper and lower bounds obtained by limit design methods. Experimental values for the strain distribution are obtained, and agree with the theory. A general discussion of the transformations of velocity fields into slip-line fields is given, and applications cited to particular problems.

E. H. Lee (Providence, R. I.).

Geiringer, Hilda. On the plane problem of a perfect plastic body. Quart. Appl. Math. **9**, 295–308 (1951).

The stress equations for the plane problem of a fully-plastic body with a general yield condition are considered. Transformations to yield linear equations are presented. Special cases for the Mises quadratic, the Tresca maximum shear, and the Mises parabolic yield limits are detailed. The characteristic relations in both the physical plane and stress plane are discussed. Solutions involving simple waves are investigated, and characteristic fields presented for the particular yield functions mentioned above. *E. H. Lee.*

Oldroyd, J. G. The motion of an elastico-viscous liquid contained between coaxial cylinders. I. Quart. J. Mech. Appl. Math. **4**, 271–282 (1951).

The author is concerned with the "stress-strain laws" governing the motion of "elastico-viscous" media. In particular, he notes [as in Oldroyd, Proc. Roy. Soc. London. Ser. A. **200**, 523–541 (1950); these Rev. **11**, 703] that the small strain-rate law $e_{ij}+\lambda_1\dot{e}_{ij}=A(\sigma_{ij}+\lambda_2\dot{\sigma}_{ij})$ must, in an Eulerian coordinate representation, contain quadratic terms in the space derivatives of velocity components and products of such derivatives with stress components. Thus, for large strain rates he finds it consistent to admit as possible laws any tensor relation containing such quadratic terms. He then notes that both qualitative and quantitative differences of behavior can be predicted for the different choices from such a family of laws. The very important implication is that small strain-rate tests cannot possibly give much insight into the large strain-rate rules of behavior if the foregoing is true. It should be noted that the author's version of the convective time derivative of a second order tensor is not consistent with that of various other authors. Note also that unless the medium under investigation has no memory whatever about strain, the time rate of change of strain is not the usual $\frac{1}{2}(u_{i,j}+u_{j,i})$. These remarks do not invalidate the above conclusion but indicate that there are several important fundamental questions which remain controversial in this field. *G. F. Carrier.*

Gassmann, F. Über die Elastizität poröser Medien. Vierteljschr. Naturforsch. Ges. Zürich **96**, 1–23 (1951).

"There are certain not strictly elastic systems which behave as perfectly elastic ones provided the stress system makes only slight fluctuations around certain mean values. . . . Such stress fluctuation take place, for example, if elastic waves run through the system. It is indeed a known fact that elastic waves, both longitudinal and shearwaves, are propagated through systems which are far from ideally elastic such as sand, gravel or clay, causing energy dissipations and permanent changes in the system which are negligible, just as if the propagation had taken place through an ideally elastic medium." The author calls this phenomenon "differential elasticity". He assumes for it the usual linear stress-strain relations but with "constants" possibly dependent on the state of deformation in which the system is at the start of the fluctuations. Assuming furthermore the solid frame of the porous material to be isotropic, the author computes the apparent elastic constants and the wave velocities of the porous system in terms of the constants of the solid frame, of the pore-filling fluid, and the porosity of the system. Generalization of the formulas to the case when the frame consists of anisotropic material is outlined. The possible influence of capillarity is discussed.

P. Neményi (Washington, D. C.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Bremmer, H. On the diffraction theory of Gaussian optics. Comm. Pure Appl. Math. **4**, 61–74 (1951).

Partendo da un oggetto piano illuminato da un'onda piana, l'autore deduce per l'ampiezza complessa nello spazio immagine l'espressione

$$u(P)=\frac{N}{4\pi^2}\int\int dO_K\varphi(K)\frac{\partial}{\partial z_P}\frac{\exp(ik_0\cdot KP)}{KP}$$
$$\times\int\int dO_Q u(Q)\exp(ik_0\cdot\overline{QQ'})\frac{\partial}{\partial z_{K'}}\frac{\exp(-ik_0\cdot Q'K)}{Q'K}$$

dove P è il punto considerato nello spazio immagine, Q e Q' due punti coniugati parassialmente, $\overline{QQ'}$ il cammino ottico da Q a Q', K un punto corrente su un piano intermedio fra il sistema ottico e il piano immagine, z la coordinata lungo l'asse ottico dO_Q e dO_K rispettivamente gli elementi di superficie nel piano oggetto e nel piano intermedio, N l'ingrandimento parassiale. Il secondo integrale doppio è esteso a tutta la superficie dell'oggetto. Il primo integrale doppio è esteso all'apertura libera di un diaframma che si suppone nel piano intermedio; il fattore $\varphi(K)$ permette di tener conto di un'eventuale lamina di fase nel piano intermedio. L'espressione data è corretta nel campo dell'ottica gaussiana e soddisfa l'equazione delle onde. L'autore estende poi la trattazione ai sistemi non gaussiani.

G. Toraldo di Francia (Firenze).

Rikitake, Tsuneji. Electromagnetic induction within the earth and its relation to the electrical state of the earth's interior. I(1). Bull. Earthquake Res. Inst. Tokyo **28**, 45–100 (1950). (English. Japanese summary)

En partant des variations du champ magnétique observées à la surface de la Terre on peut séparer les actions primaires extérieures des courants induits qu'elles produisent, et obtenir des renseignements sur la conductibilité interne du Globe. L'auteur adapte des méthodes mathématiques déjà élaborées, notamment par Price, à un grand nombre de cas: Le Globe peut comprendre une sphère de conductivité uniforme entourée de deux enveloppes moins conductrices. Les variations considérées sont la variation diurne moyenne, la variation diurne des jours perturbés, la variation au cours d'un orage magnétique moyen, la variation au cours d'une baie magnétique. *J. Coulomb.*

Rikitake, Tsuneji. Electromagnetic induction within the earth and its relation to the electrical state of the earth's interior. I(2). Bull. Earthquake Res. Inst. Tokyo **28**, 219–262 (1950). (English. Japanese summary)

Poursuivant le travail analysé ci-dessus, l'auteur considère les "crochets" et les "débuts brusques" magnétiques. Dans le cas des premiers il remplace l'induction électromagnétique dans une sphère par l'induction sous un plan; il estime l'erreur commise. *J. Coulomb* (Paris).

Rikitake, Tsuneji. Electromagnetic induction within the earth and its relation to the electrical state of the earth's interior. II. Bull. Earthquake Res. Inst. Tokyo **28**, 263–283 (1950). (English. Japanese summary)

La première partie de ce travail [voir les deux analyses ci-dessus] suggère l'essai d'un modèle de Terre où la conductivité soit constante dans une couche, sous laquelle elle croît comme la puissance $-n$ de la distance au centre. L'auteur reprend avec ce modèle les problèmes qu'il a traités précédemment. Il suit encore les méthodes de Price, mais avec de nombreuses approximations. Il trouve n de l'ordre de 10. *J. Coulomb* (Paris).

* **De Vogelaere, René. Une nouvelle famille d'orbites périodiques dans le problème de Störmer: Les ovales.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 170–171. University of Toronto Press, Toronto, 1951. $6.00.

Viglin, A. S. On the averaging of the microscopic equations of the electromagnetic field in 4-dimensional form. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 795–802 (1951). (Russian)

In classical electromagnetic theory the Maxwell equations in the presence of solid matter may be written in two ways. Either the matter is represented as a collection of point charges, and the Maxwell equations describe the fields in the vacuum between the charges (microscopic view-point). Or alternatively the matter is represented as a continuous distribution of electric and magnetic polarization, and the equations describe the fields and the polarization phenomenologically (macroscopic view-point). The author here shows how to obtain the macroscopic from the microscopic equations in a relativistically invariant way.

F. J. Dyson (Ithaca, N. Y.).

de Possel, René, et Pouget-Michel, Colette. Sur le principe de Huyghens pour une onde électromagnétique. C. R. Acad. Sci. Paris **232**, 1819–1821 (1951).

The authors give an integral formula expressing an electromagnetic wave-function (not necessarily sinusoidal) in terms of surface distributions of electric and magnetic dipoles, without the introduction of surface charges and currents. They verify that their formula agrees with classical formulae of F. Kottler [Ann. Physik (4) **71**(376), 457–508 (1923)] and others. *F. V. Atkinson* (Ibadan).

Schelkunoff, S. A. Kirchhoff's formula, its vector analogue, and other field equivalence theorems. Comm. Pure Appl. Math. **4**, 43–59 (1951).

Kirchhoff's formula, as used here, is the mathematical interpretation of Huygens' principle in the theory of wave propagation. It is very useful in the treatment of diffraction problems. The case of a source inside a perfectly rigid horn is considered especially. In Kirchhoff's formula there occurs an integral extended over a closed surface S containing the horn. This closed surface may be chosen to consist of an "aperture surface" S_a together with the exterior surface of the horn; the integrand depends on a certain Green's function. However, approximations are necessary. It is proved here that no improvement of the approximation is to be obtained by using Kirchhoff's formula repeatedly.

For the case of electromagnetic waves the author [Physical Rev. (2) **56**, 308–316 (1939)] has deduced another theorem which he calls the "induction theorem" in which the surface of integration is solely the aperture surface S_a. In this theorem there appears a distribution of virtual sources over the aperture which in the presence of the horn produces the correct external field of the truncated horn and the field returned into the horn by the truncation. Kirchhoff's formula involves scalar wave-functions, while the induction theorem in its original form contains vector wave-functions. In this paper the author gives an analogue of the induction theorem for scalar wave-functions. The former form is suited for the electromagnetic case, the latter for the acoustic case. Further, the author discusses various field equivalence theorems which express the identity, in certain specified regions, of the field produced by the given sources and the field produced by appropriate sources on the boundary of the regions. *H. Bremekamp* (Delft).

Andrejewski, W. Strenge Theorie der Beugung ebener elektromagnetischer Wellen an der vollkommen leitenden Kreisscheibe und an der kreisförmigen Öffnung im vollkommen leitenden ebenen Schirm. Numerische Ergebnisse. Naturwissenschaften **38**, 406–407 (1951).

Keller, Joseph B., and Blank, Albert. Diffraction and reflection of pulses by wedges and corners. Comm. Pure Appl. Math. **4**, 75–94 (1951).

Almost identical with an earlier report [New York University, Washington Square College, Mathematics Research

Group, Research Rep. No. **EM-21** (1950); these Rev. **12**, 564]. *C. J. Bouwkamp* (Eindhoven).

Levine, Harold, and Schwinger, Julian. On the theory of electromagnetic wave diffraction by an aperture in an infinite plane conducting screen. Comm. Pure Appl. Math. **3**, 355–391 (1950).

After briefly surveying theoretical methods available for three-dimensional diffraction problems, the authors stress the need for approximation procedures, accurate in a large frequency range. This paper, a sequel to previous ones concerned with scalar problems [Physical Rev. (2) **74**, 958–974 (1948); **75**, 1423–1432 (1949); these Rev. **10**, 221, 764], describes the nature of variational principles for obtaining some of the desired information. The present approach is exactly similar to that for the scalar problems studied in the papers cited. Two different variational principles are developed, based on the rigorous field representation by means of surface integrals and dyadic Green's functions [cf. Schwinger, Massachusetts Institute of Technology Radiation Laboratory, report **43-34** (1943)], with special emphasis on the transmission cross-section of the aperture. The variational technique is illustrated at the diffraction of a normally incident plane-polarized electromagnetic wave through a circular aperture. Numerical results for the transmission cross-section are compared with those found from the Kirchhoff and Rayleigh approximations. As added in proof, these results require important qualification in view of recent work by the reviewer [Philips Research Rep. **5**, 321–332, 401–422 (1950); these Rev. **12**, 774]. The appropriate modifications are included. *C. J. Bouwkamp.*

Schachenmeier, Richard. Die Beugung elektromagnetischer Wellen an der Erde im optisch inhomogenen Medium. Arch. Elektr. Übertragung **5**, 267–272 (1951).

The discussion of experimental data obtained from tropospheric radio wave propagation leads to the problem mentioned in the title. The author states that this problem is new (May 1949). His approach is based on geometrical optics and differential geometry. Eckersley's concept of modified curvature of the earth for slightly inhomogeneous stratified media is discussed. The paper is not so much conclusive as suggestive. *C. J. Bouwkamp* (Eindhoven).

Twersky, Victor. On the nonspecular reflection of electromagnetic waves. J. Appl. Phys. **22**, 825–835 (1951).

The technique employed by the author is essentially an extension of Rayleigh's problem of the boss. Exact solutions obtained for the reflection problem of a plane wave of arbitrary polarization incident at an arbitrary angle on either a semicylindrical or hemispherical boss on an infinite plane are extended subject to the single scattering hypothesis to obtain the far field solutions for certain patterned and uniform random distributions of bosses of radii small compared with the wave length. The different solutions obtained in the above cases are then discussed and compared. *M. J. O. Strutt* (Zurich).

Twersky, Victor. On the nonspecular reflection of sound from planes with absorbent bosses. J. Acoust. Soc. Amer. **23**, 336–338 (1951).

This paper is an extension of the one reviewed above dealing with the reflection of plane waves from a rigid non-absorbing non-porous surface with cylindrical and hemispherical bosses. In the present paper the impedance of the bosses is arbitrary and that of the plane is either 0 or ∞. He starts with the semicylindrical boss and then proceeds with the hemispherical boss. In each case an expression is derived for the reflection coefficient. *M. J. O. Strutt* (Zurich).

Friedman, Bernard. The dipole field in an inhomogeneous atmosphere. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-28**, iii+56 pp. (1951).

The author presents a mathematical theory of radio wave propagation above a spherical earth for stratified media with an arbitrary radial variation of the refractive index. The theory corresponding to a radial magnetic Hertzian dipole used as the source of radiation is worked out in detail; appropriate changes to be made for the case of an electric dipole are indicated. The transformation used by Watson [cf. H. Bremmer, Terrestrial Radio Waves, Elsevier, New York-Amsterdam-London-Brussels, 1949; these Rev. **11**, 295; A Sommerfeld, Partielle Differentialgleichungen der Physik, Dieterich, Wiesbaden, 1947, pp. 216–227, 282–291; these Rev. **10**, 195] is extended to the general case. It is shown that the Hertzian potential for a non-uniform atmosphere can be obtained from that for a constant atmosphere by making suitable substitutions, which until now was known for certain special refractive-index variations only. Some mathematical questions concerning the continuous spectrum are examined in detail. However, the reviewer is unable to follow the author's argument in appendix I, pp. 50–51. The last integral, in which the factor $1/\theta$ should be transferred under the sign of integration, is actually divergent. The peculiarities at $\theta=0$, where $P_\nu(-\cos\theta)$ becomes infinite, are not apparent from the author's work. In fact, the reviewer would not be surprised at all if someone were able to prove that the residue series does not converge when $0\leqq\theta<\frac{1}{2}\pi$. *C. J. Bouwkamp* (Eindhoven).

Marcuvitz, N., and Schwinger, J. On the representation of the electric and magnetic fields produced by currents and discontinuities in wave guides. I. J. Appl. Phys. **22**, 806–819 (1951).

This first paper (which "has lain dormant for a number of years") of a planned series of two may now serve as an introduction to the mathematical theory of equivalent-circuit representation of waveguide discontinuities, a subject to which both authors have contributed materially.
C. J. Bouwkamp (Eindhoven).

Levine, Harold. The wavelength of a spherical resonator with a circular aperture. J. Acoust. Soc. Amer. **23**, 307–311 (1951).

The fundamental wavelength λ of a spherical resonator (radius R) with a circular aperture (radius a) is calculated by use of Schwinger's variational technique. The result is

$$\lambda/2\pi=R(2\pi R/3a)^{\frac{1}{2}}[1-0.14324(a/R)+0.08455(a/R)^2],$$

which extends and corrects the earlier results of Rayleigh [Proc. Roy. Soc. London. Ser. A. **92**, 265–275 (1916)] and White [ibid. **92**, 549–555 (1916)] respectively. A procedure for approximating to the fundamental wavelength of an arbitrary Helmholtz resonator is described.
C. J. Bouwkamp (Eindhoven).

Sollfrey, William. Wave propagation on helical wires. J. Appl. Phys. **22**, 905–910 (1951).

Abstract based on the author's summary: Previous researches on the helix have made simplifying approximations about the nature of the boundary conditions. In this paper

the field equations and boundary conditions are formulated exactly, using a non-orthogonal helical coordinate system. They are then solved by a perturbation procedure in terms of an expansion in powers of the ratio (thickness of wire)/(distance between turns) of which only the first is actually retained. The result shows a principal mode which propagates at free-space velocity of light in the direction of the wire. The characteristics of this mode are studied and compared with experiment. *C. J. Bouwkamp.*

Graffi, Dario. Sulla propagazione delle onde di tipo elettrico o magnetico in una guida a sezione circolare riempita da un dielettrico eterogeneo. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **7** (1949–50), 123–132 (1951).

Quantum Mechanics

Dirac, P. A. M. Generalized Hamiltonian dynamics. Canadian J. Math. **2**, 129–148 (1950).

Describes a generalized form of Hamiltonian dynamics, applicable when the moments are not independent functions of the velocities, which can still be used for quantization purposes and which is well-suited for a relativistic description of dynamical processes. In the usual notation, the elimination of the velocities from the equations $p_n=\partial L/\partial \dot{q}_n$ leads to a set of independent equations $\varphi_m(q, p)=0$. A Hamiltonian function $\mathfrak{H}(p, q)$ and a set of new variables v_m are introduced in terms of which the equations of motion can be written

$$\dot{q}_n=\frac{\partial \mathfrak{H}}{\partial p_n}+v_m\frac{\partial \varphi_m}{\partial p_n}, \quad \dot{p}_n=-\frac{\partial \mathfrak{H}}{\partial q_n}-v_m\frac{\partial \varphi_m}{\partial q_n},$$

the fundamental variables p, q, v being called the Hamiltonian variables. The theory takes a specially simple form when the Lagrangian is homogeneous of the first degree in the velocities, in which case $\mathfrak{H}$ is zero. Poisson brackets may be introduced and the equations of motion then written $\dot{g}=v_m[g, \varphi_m]$, where g is any function of the p's and q's; and it is shown that in this last equation the φ's can be reduced to a minimum set of "first-class" φ's, whose Poisson brackets with each other vanish. (When the classical dynamical system is made homogeneous in this way there is only one φ.) In applying the theory to relativistic dynamics, we may express our equations (i) in instant-form requiring four first-class φ's; or (ii) in point-form also requiring four φ's; or (iii) in front-form requiring three φ's; or (iv) in general surface-form with infinitely many φ's. To quantize the system we set up as usual a scheme of linear operators corresponding to the dynamical variables q and p (there are no operators corresponding to the v's) according to the principles: (a) Poisson bracket relations between the classical variables correspond to commutation relations between the operators; (b) equations $X(q, p)=0$ between the classical variables correspond to linear conditions $X\psi=0$ on the wave function ψ. The equation $\varphi=0$ of this generalized dynamics leads by (b) to the Schrödinger equations $\varphi\psi=0$, and there are as many Schrödinger equations as there are first-class φ's required for the dynamical system.

A. J. McConnell (Dublin).

Dirac, P. A. M. The Hamiltonian form of field dynamics. Canadian J. Math. **3**, 1–23 (1951).

If dynamical theory is expressed in terms of changes which occur as one proceeds from one space-like surface to another, the surface variables must be treated on the same footing as the dynamical coordinates. The general solution of the equations of motion must then involve some arbitrary functions or functionals. This necessitates a generalization of Hamiltonian dynamics developed earlier [see the preceding review] and applied here to the computation of Poisson bracket relations involving the surface variables and their conjugate momenta, to changes of parameterization of the surface, to the passage from the Lagrangian to the Hamiltonian formalism, and to particular fields.

H. C. Corben (Pittsburgh, Pa.).

Dirac, P. A. M. A new meaning for gauge transformations in electrodynamics. Nuovo Cimento (9) **7**, 925–938 (1950).

This paper contains an application of the author's generalized theory of Hamiltonian dynamics announced in August 1949 at the Canadian Mathematical Seminar [see the two preceding reviews]. The author begins by putting the theory of the electromagnetic field in a Hamiltonian form but instead of employing the usual covariant components of the four-potential as the field variables, he uses its components tangential and normal to a one parameter family of space-like surfaces. Spinless electrons are then introduced and the familiar interaction Lagrangian is assumed. A complicated expression for the Hamiltonian is obtained. The space-like surfaces are then so specialized, and the allowed gauge transformations are so linked with these surfaces as to simplify the Hamiltonian somewhat. Corresponding to interaction between the electrons and the field a peculiar new term appears in the Hamiltonian which contains the factor e^{-2}. From this it immediately follows that the usual procedure of seeking an approximate solution by means of a series in powers of e will be useless in this theory. Another novelty of the theory is that the supplementary condition is abandoned so permitting arbitrary gauge transformations. This means that the longitudinal field variables are not eliminated and so the relativistic appearance of the theory is not spoiled. *A. J. Coleman* (Toronto, Ont.).

Pirani, F. A. E., and Schild, A. On the quantization of Einstein's gravitational field equations. Physical Rev. (2) **79**, 986–991 (1950).

This paper is based on the work of P. Weiss [Proc. Roy. Soc. London. Ser. A. **169**, 102–119, 119–133 (1938)] and of Dirac [see the third preceding review]; it is also related to the paper of Bergmann and Brunings [Rev. Modern Physics **21**, 480–487 (1949); these Rev. **11**, 299]. The authors begin with a Lagrangean density function characterizing the field and, by the use of parameters associated with a family of three-dimensional surfaces, set up a canonical formalism involving Poisson brackets. The quantization is carried out by replacing the Poisson brackets by commutators in the usual manner. This procedure is applied to a gravitational field and is also extended to combined gravitational and electromagnetic fields. *N. Rosen* (Chapel Hill, N. C.).

Kwal, Bernard. Les méthodes covariantes de mécanique analytique en théorie générale des champs, linéaires ou non. J. Phys. Radium (8) **12**, 534–542 (1951).

The author shows that a relativistically invariant form of the Hamiltonian equations can be obtained for a complex field the Lagrangian of which depends on the field quantities q^k, their first order covariant derivatives and their complex conjugates. The cases in which q^k is a tensor or a spinor are treated separately. It is well known that equivalent or more

general treatments of this question have recently been given by other authors, for example, Weiss [Proc. Roy. Soc. London. Ser. A. 169, 102–119, 119–133 (1938)] and de Wet [Proc. Cambridge Philos. Soc. 44, 546–559 (1947); these Rev. 10, 91]. *A. J. Coleman* (Toronto, Ont.).

Thermodynamics, Statistical Mechanics

*de Groot, S. R. **Thermodynamics of Irreversible Processes.** North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York, N. Y., 1951. xvi+242 pp. $5.00.

This book presents a systematic treatment of thermal, electrical, and material fluxes together with their associated "forces" and "cross-effects", account being taken of such factors as viscosity and chemical reactions. The methodology consists essentially of the following steps: (1) Assume that the fluxes and forces are connected by a linear system of relations, these fluxes and forces being characterized by the bilinear form for the entropy production (or in some equivalent way). (2) Assume Onsager's relations among the coefficients in this linear system. (3) Determine the physical interpretations of these coefficients by considering significant special cases of the linear system. The linear system then expresses the fundamental phenomenological laws of the physical system considered. The author indicates the domain of thermodynamic phenomena that is susceptible to this treatment, he gives a derivation of Onsager's relations (based on microscopic reversibility), and he applies the theory to stationary states. However, no physical "explanation" is given as to why "any force can give rise to any flow", so that the treatment must be considered essentially empirical. Nevertheless, as an empirical treatment, the method is remarkably successful. *C. C. Torrance.*

Meissner, Hans. Zur Theorie der Korrelation der Dichteschwankungen in realen Gasen. Z. Physik 130, 202–213 (1951).

The author derives anew and discusses some little known results of Ornstein and Zernike [Nederl. Akad. Wetensch., Proc. 17, 793–806 (1914) and other papers] and Didlaukies [Ann. Physik (5) 5(397), 205–243 (1930)] concerning the correlation between values of the density at different points in a gas in equilibrium. He states that for any gas these fluctuations are qualitatively those of the van der Waals gas. The analysis is of the heuristic type, employing differential volume elements "in which there is at most one molecule," etc. *C. Truesdell* (Bloomington, Ind.).

Cohen, Robert S., Spitzer, Lyman, Jr., and Routly, Paul McR. The electrical conductivity of an ionized gas. Physical Rev. (2) 80, 230–238 (1950).

In the theory of nonuniform gases [as expounded, for example, by S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Cambridge, 1939; these Rev. 1, 187] the interaction term in the Boltzmann equation is expressed in terms of individual close binary encounters. When the force between two particles varies as the inverse square of their mutual separation this is no longer satisfactory on account of the cumulative effect of relatively distant encounters which becomes important. The authors follow an earlier suggestion of the reviewer [Astrophys. J. 97, 255–262, 263–273 (1943); these Rev. 4, 260] that the proper way to treat the problem under these circumstances is by using an equation of the Fokker-Planck type as in the theory of Brownian motion. On this picture the interaction term should consist of two terms: a term

$$J(f_r f_s)=\int_0^\infty dv \int_0^{2\pi} d\epsilon \int_0^{b_0} dbgb(f_r f_s - f_r' f_s')$$

where g is the relative velocity $|\mathbf{v}_r-\mathbf{v}_s|$ between two particles, one of type r and one of type s, f_r is the velocity distribution function for particles of type r, f_r' and f_s' are the values of f_r and f_s for velocities such that a particle of type r will be left after the encounter within volume element $dv_r dv_\theta dv_\phi$ considered, b is the impact parameter, b_0 the maximum value of b for which the effect of a single encounter may be considered appreciable, ϵ the angle between the orbital plane and the plane containing the two particles before the encounter; and a "diffusion term"

$$K(f_r f_s)=\sum_i \frac{\partial}{\partial v_i}(f_r\langle \Delta v_{i,s}\rangle)-\tfrac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial v_i \partial v_j}(f_r\langle \Delta v_{i,s}\Delta v_{j,s}\rangle),$$

where for any quantity x, $\langle x_s\rangle dt$ represents the mean value of x resulting from all encounters with particles of type s during the time interval dt. For inverse square forces these averages diverge logarithmically when integrating over b. In the earlier discussions of this subject the integration was cut off at $b_{\max}$ equal to the average distance between the particles. But the authors present convincing arguments for taking $b_{\max}$ to be equal to the so-called Debye distance $h=[kT/4\pi n_e e^2(1+Z_i)]^{\frac{1}{2}}$ (where Z_i is the average charge of the ions and the rest of the symbols have their usual meanings) at which the electron-ion plasma shields a particular charge.

For the problem of electrical conductivity, the solution for the distribution function is sought in the form

$$f=f^{(0)}(v)[1+D(jv)\cos\theta] \tag{1}$$

where $f^{(0)}(v)=(n_e j^3/\pi^{\frac{1}{2}})e^{-j^2v^2}$, denotes the Maxwell distribution and θ is the angle between the directions of $\mathbf{v}$ and of the applied electric field $\mathbf{E}$. For f of the form (1), it is shown that the interaction term arising from the electron-electron encounters is, for example, given by

$$\begin{aligned}(2)\quad K(ff)=&\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 f\left\{\langle\Delta_t\rangle+\frac{1}{v}\langle\Delta_v^2\rangle-\frac{1}{2v^2}\frac{\partial}{\partial v}(v^2\langle\Delta_t^2\rangle)\right\}\right]\\&+\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\left\{\langle\Delta_v\rangle-\frac{1}{2v}\frac{\partial}{\partial\theta}\langle\Delta_v^2\rangle\right\}\right]\\&-\frac{1}{2v^2}\frac{\partial}{\partial v}\left[v^2\langle\Delta_t^2\rangle\frac{\partial f}{\partial v}\right]-\frac{1}{2v^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\langle\Delta_v^2\rangle\frac{\partial f}{\partial\theta}\right]\\&-\frac{1}{v^2\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\langle\Delta_t\Delta_v\rangle+\frac{\partial}{\partial v}(vf\sin\theta\langle\Delta_t\Delta_v\rangle)\right]\end{aligned}$$

where Δ_t, Δ_v and Δ_r are the increments in the velocity due to a binary encounter along the direction of $\mathbf{v}$ and in two directions at right angles to it. And the Boltzmann equation to be solved is

$$(2j^2ef^{(0)}/m)Ev\cos\theta+K(ff_p)+K(ff)=0. \tag{3}$$

The term $K(ff_p)$ arising from the electron-proton interactions is given by (2) if the averaging is performed over the distribution f_p instead of f.

In the actual evaluation of $K(ff)$ and $K(ff_p)$ the last two terms in (2) were neglected. All the other terms were

evaluated. The term $K(ff_p)$ is particularly simple since the protons may be assumed to be at rest and only one term survives. The terms $\langle\Delta_i^2\rangle$, $\langle\Delta_e^2\rangle$ and $\langle\Delta_i\rangle$ occurring in the expression for $K(ff)$ can be expressed in the forms $\langle\Delta_i^2\rangle=p_0+p_1\cos\theta$, $\langle\Delta_e^2\rangle=q_0+q_1\cos\theta$ and $\langle\Delta_i\rangle=r_0+r_1\cos\theta$ where p_0, q_0 and r_0 are certain functions of x involving the error function and p_1, q_1 and r_1 are functions of x and involve the unknown function $D(x)$ through integrals of the form $I_n(x)=\int_0^x D(y)y^n e^{-y^2}dy$. The equation to be solved is finally reduced to the form

$$D''(x)+P(x)D'(x)+Q(x)D(x)=R(x)+S(x) \tag{4}$$

where P, Q and R are certain known functions of x and $S(x)$ involves the unknown function $D(x)$ through the integrals I_0, I_3 and I_5. The solution of equation (4) is sought which has a convergent series expansion in ascending powers of x for $x\to 0$ and descending powers of x as $x\to\infty$. Apparently the equation is very unstable for $x\to 0$ and $x\to\infty$ and the authors describe the difficulties they had to overcome in obtaining the required solution. But a solution was found and is tabulated. The end result of the paper is the evaluation of the electrical conductivity σ. The expression found for this is $\sigma=1.470[2m/3\pi^{\frac{1}{2}}e^2j^2\log(qC^2)]$ where C^2 is the root mean square velocity and $q=(m/2e^2Z_i)[kT/\pi n_e(1+Z_i)]^{\frac{1}{2}}$. [The number 1.470 is essentially $I_5(\infty)$.] The formula for σ which the authors derive gives values for the conductivity which are about 60 per cent of the values found by Cowling on the more conventional treatment of the problem.

S. Chandrasekhar (Williams Bay, Wis.).

Wakefield, A. J. Statistics of the simple cubic lattice. Proc. Cambridge Philos. Soc. **47**, 419–435 (1951).

The author describes the partition function for a ferromagnet involving an external field and the interaction of nearest neighbors and also the partition function for a "regular solution" consisting of two kinds of atoms arranged on a cubical lattice. The latter partition function is somewhat more complex than the former but has a certain similarity in form and the author describes an additional mathematical procedure which will permit one to refer the "regular solution" problem to that of the ferromagnet. A series expansion for the ferromagnetic partition function valid for low temperatures is stated. Using a method ascribed to Domb, the author obtains, subject to a convergence assumption, two rearrangements of this series which are applicable in the high temperatures case. In addition, the author gives a direct geometrical derivation of the high temperature expression in an interesting way, based on an inversion due to Opechowski. (There is a mathematical appendix which does much to unify this discussion.) If the sign of the energy bond between neighboring atoms is reversed, new phenomena including anti-ferromagnetism, may be treated by the same series methods. The same high temperature expressions are valid but different low temperature expansions appear depending on the strength of the external field.

F. J. Murray (New York, N. Y.).

Tanaka, Tomoyasu, Katsumori, Hiroshi, and Toshima, Soichiro. On the theory of cooperative phenomena. Progress Theoret. Physics **6**, 17–26 (1951).

By direct counting, low temperature expansions for the partition function $(f^N)^{1/N}$ are obtained for Ising model lattices in the following cases: face centered cubic, 2-dimensional triangular, simple cubic lattice, body centered cubic lattice. The result is applied to various physical phenomena, for instance, spontaneous magnetization at low temperatures, problems of solution, properties of liquids, layer adsorption.

F. J. Murray (New York, N. Y.).

Montroll, E. W., and Berlin, T. H. An analytical approach to the Ising problem. Comm. Pure Appl. Math. **4**, 23–30 (1951).

The authors propose a fundamentally new approach to the problem of the evaluation of the partition function $f(K)$ of a lattice which consists of N points arranged in a pattern so that each point has a relatively small number of nearest neighbors. Each point has associated a spin σ_j having the values ± 1. Let a_{ij} be one or zero accordingly as the ith point and the jth point are nearest neighbors or not. Then $f(K)=\sum(\exp(K\sum_{j,k}a_{jk}\sigma_j\sigma_k))$ where the summation is over all N tuples $\sigma_1,\cdots,\sigma_N$ in which σ_j has the value ± 1. The basic maneuver here is to replace the summation relative to the discrete variable σ_j by a double Fourier integral involving the Dirac delta function. Thus, the N-fold summation is, in the first instance, replaced by two iterated N-fold integrations. But the innermost integration can be carried out explicitly and one obtains an N-fold integral expression for $f(K)$ which is essentially the Fourier transform of $D(y_1,\cdots,y_N)^{-\frac{1}{2}}$ where D is the Nth order determinant whose diagonal elements are $iy_k+b_k+\epsilon_k$ and off-diagonal elements are $-a_{jk}$. Here ϵ_k is the number of nearest neighbors of the kth point. The y_k's are the variables of integration in the N-fold integral and it is convenient to replace each such integration by an integration along a line parallel to the imaginary axis. For this integral, the determinant D is given explicitly in a number of cases including the one, two and three-dimensional rectangular lattices. Finally, $f(K)$ is completely evaluated in the one-dimensional case.

F. J. Murray (New York, N. Y.).

Hoffmann, T. A., and Kónya, A. Some investigations in the field of the theory of solids. I. Linear chain of similar atoms. Acta. Phys. Acad. Sci. Hungaricae **1**, 5–35 (1951). (English. Russian summary)

The solution of the quantum mechanical problem of a linear chain of n like atoms with atomic wave functions $\psi_1,\psi_2,\cdots,\psi_n$ is carried out by means of a linear combination of the atomic orbitals $\psi=c_1\psi_1+c_2\psi_2+\cdots+c_n\psi_n$ following molecular orbital methods in chemistry. The energy states of a finite chain and an infinite chain are calculated together with the wave function. End effects or boundary conditions on the chain lead to calculation of work function and surface dipole moments. It is proposed that the quantum mechanics of solids may be developed by the methods of quantum chemistry and by regarding a crystalline solid as a very large chain molecule.

R. Truell (Providence, R. I.).